

On instanton contributions to anomalous dimensions in $\mathcal{N}=4$ supersymmetric Yang–Mills theory

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Abstract

Instanton contributions to the anomalous dimensions of gauge-invariant composite operators in the $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang–Mills theory are studied in the one-instanton sector. Independent sets of scalar operators of bare dimension $\Delta_0 \leq 5$ are constructed in all the allowed representations of the $SU(4)$ R-symmetry group and their two-point functions are computed in the semiclassical approximation. Analysing the moduli space integrals the sectors in which the scaling dimensions receive non-perturbative contributions are identified. The requirement that the integrations over the fermionic collective coordinates which arise in the instanton background are saturated leads to non-renormalisation properties for a large class of operators. Instanton-induced corrections to the scaling dimensions are found only for dimension 4 $SU(4)$ singlets and for dimension 5 operators in the **6** of $SU(4)$. In many cases the non-renormalisation results are argued to be specific to operators of small dimension, but for some special sectors it is shown that they are valid for arbitrary dimension. Comments are also made on the implications of the results on the form of the instanton contributions to the dilation operator of the theory and on the possibility of realising its action on the instanton moduli space.

Contents

1	Introduction and summary of results	1
2	One-instanton contributions to correlation functions in $\mathcal{N}=4$ SYM	7
3	The $\mathcal{N}=4$ instanton supermultiplet	11
4	Instanton induced two-point functions and anomalous dimensions	14
4.1	Two-point functions in the one-instanton sector	15
4.2	Anomalous dimensions	16
5	Non-perturbative contributions to anomalous dimensions of scalar operators	18
5.1	Dimension 2 scalar operators	19
5.2	Dimension 3 scalar operators	21
5.2.1	$\Delta_0 = 3, [0, 1, 0]$	21
5.2.2	$\Delta_0 = 3, [2, 0, 0]$ and $[0, 0, 2]$	23
5.2.3	$\Delta_0 = 3, [0, 3, 0]$	24
5.2.4	$\Delta_0 = 3, [1, 1, 1]$	25
5.2.5	$\Delta_0 = 3, [0, 1, 0]$ revisited	25
5.3	Dimension 4 scalar operators	26
5.3.1	$\Delta_0 = 4, [0, 0, 0]$	27
5.3.2	$\Delta_0 = 4, [1, 0, 1]$	32
5.3.3	$\Delta_0 = 4, [0, 2, 0]$	34
5.3.4	$\Delta_0 = 4, [2, 1, 0]$ and $[0, 1, 2]$	35
5.3.5	$\Delta_0 = 4, [2, 0, 2]$	37
5.3.6	$\Delta_0 = 4, [0, 4, 0]$	39
5.3.7	$\Delta_0 = 4, [1, 2, 1]$	40
5.4	Dimension 5 scalar operators	40
5.4.1	$\Delta_0 = 5, [0, 1, 0]$	42
5.4.2	$\Delta_0 = 5, [2, 0, 0]$ and $[0, 0, 2]$	47
5.4.3	$\Delta_0 = 5, [0, 3, 0]$	48
5.4.4	$\Delta_0 = 5, [1, 1, 1], [3, 0, 1], [1, 0, 3], [2, 2, 0]$ and $[0, 2, 2]$	49
5.4.5	$\Delta_0 = 5, [0, 5, 0]$	49
5.4.6	$\Delta_0 = 5, [2, 1, 2]$ and $[1, 3, 1]$	50
6	Discussion and conclusions	50
A	Conventions and useful relations	56
B	One-instanton in $\mathcal{N}=4$ SYM	57
C	OPE analysis of a four-point function	63
	References	66

1 Introduction and summary of results

Recent progress in the study of the interconnections between string and gauge theories has revived the interest for the computation of anomalous dimensions in $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) theory. In a conformal field the spectrum of scaling dimensions, together with the set of OPE coefficients for a complete basis of operators, constitutes the fundamental physical information. In the original formulation of the AdS/CFT correspondence [1–3], which in its best understood form relates the $\mathcal{N}=4$ SYM theory with

SU(N) gauge group to type IIB string theory in $\text{AdS}_5 \times S^5$, the possibility of comparing this physical information with the dual string theory quantities is limited by the present inability of quantising strings in the relevant background. In the correspondence scaling dimensions of gauge-invariant composite operators are related to the energy of the dual string/supergravity states. The parameters of the string and gauge theories are related by

$$4\pi g_s = g_{\text{YM}}^2, \quad \left(\frac{L^2}{\alpha'}\right)^2 = g_{\text{YM}}^2 N \equiv \lambda, \quad (1)$$

which imply the strong/weak coupling nature of the duality: in the limit in which classical supergravity is a good approximation, which is the only limit in which string theory in $\text{AdS}_5 \times S^5$ is under control, the gauge theory is strongly coupled. This makes a direct comparison of quantities calculated at weak coupling in field theory with quantities computed in supergravity impossible. This is true in particular for the comparison of field theory scaling dimensions and energies of supergravity states.

In the early days of the correspondence anomalous dimensions first emerged in connection with logarithmic singularities in the short distance limit of four-point functions [4]. The singularities in the so called conformal partial wave expansion of four-point amplitudes in supergravity [5] are interpreted as due to quantum corrections to the mass of states exchanged in intermediate channels. A similar analysis of the operator product expansion (OPE) of the dual four-point correlation functions in field theory displays the same type of singularities [6], which in turn are attributed to anomalous dimensions of operators entering the OPE [4]. Although the physical mechanisms which are responsible for the observed divergences are correctly identified, a quantitative comparison is impossible because the two calculations have different regimes of validity. This is a general feature and until recently, in spite of what appear to be non-trivial tests of the duality, relatively little insight into truly dynamical aspects had been obtained.

An important step forward was made with the proposal of [7] where a limit of the AdS/CFT correspondence was identified in which the comparison between string theory and gauge theory can be carried on avoiding the strong/weak coupling problem. On the string side one considers a maximally supersymmetric plane-wave background obtained as Penrose limit of $\text{AdS}_5 \times S^5$ [8]. Although there is a nontrivial $\text{R} \otimes \text{R}$ condensate, the quantisation of string theory in this background is feasible because the world sheet theory in the light-cone gauge reduces to a free massive model [9]. The dual of string theory in this background was identified in [7] as a particular sector of $\mathcal{N}=4$ SYM made of operators of large scaling dimension, Δ , and with large value of the charge, J , with respect to one of the R-symmetry generators. The holographic nature of the duality in this limit, referred to as BMN limit, is not well understood, but a prescription for comparing the mass spectrum of the string with the spectrum of dimensions of gauge theory operators has been given. Using this prescription a remarkable agreement between string and field theory results has been obtained in perturbation theory [10–12]. What is particularly interesting in the BMN correspondence is that the comparison can be extended beyond supergravity to include massive string modes and moreover it can be carried on in a quantitative way. This is because the duality arises in a double scaling limit in which the rank of the gauge group is sent to infinity and at the same time $J \rightarrow \infty$, in such a way that J^2/N is kept fixed. In this limit and in the BMN sector of the gauge theory the

relevant parameters are $\lambda' = g_{\text{YM}}^2 N/J^2$ and $g_2 = J^2/N$ [13, 14]. The dual string theory can also be formulated in terms of the same parameters and in the BMN limit on both sides one can consider an expansion for small values of λ' and g_2 .

Another new idea, which was suggested in [15] extending considerations in [16], has lead to further exciting developments. What was noticed in these papers is that in certain limits, in which some quantum numbers take large values, the semiclassical description of the $\text{AdS}_5 \times S^5$ string sigma model can represent a good approximation. These concepts have been employed in several recent papers [17–25] in which solitonic solutions of the sigma-model, corresponding to strings rotating in $\text{AdS}_5 \times S^5$, have been put in correspondence with gauge theory operators. The comparison of the energy of such solitonic string configurations with the anomalous dimensions of the dual operators has opened the possibility of new quantitative tests of the AdS/CFT correspondence, which appear to have a dynamical nature. Very accurate comparisons of the spectra on the two sides of the correspondence have been performed. An important rôle in these calculations in the gauge theory is played by the observation that the problem of computing anomalous dimensions can be rephrased as an eigenvalue problem for the dilation operator of the theory [26–29]. When the problem is recast in these terms the structure of an integrable system emerges in the planar limit [27, 30]. This allows to apply the techniques of the Bethe ansatz to the computation of anomalous dimensions. The integrability emerging in the planar limit of the $\mathcal{N}=4$ theory appears to play a crucial rôle in these recent tests of the AdS/CFT duality and is believed to have a counterpart in the integrability of the free string theory in $\text{AdS}_5 \times S^5$ [31–33].

All the recent developments mentioned here concern the perturbative sector of the gauge theory and very little has been done in the study of quantum corrections to anomalous dimensions beyond perturbation theory. One of the purposes of this paper is to initiate a systematic analysis of the non-perturbative instanton-induced corrections to the spectrum of anomalous dimensions in $\mathcal{N}=4$ SYM. A fundamental motivation for this comes from the observation that instantons are expected to play a special rôle in $\mathcal{N}=4$ SYM due the S-duality of the theory.

The study of instanton effects in the AdS/CFT correspondence provides an example of a situation in which the agreement between string and gauge theory results appears highly non-trivial already at the level of supergravity. Here the duality relates effects of instantons in $\mathcal{N}=4$ SYM and D-instantons in type IIB string theory [34]. Instanton corrections to correlation functions have a counterpart in contributions to string/supergravity amplitudes induced by the presence of D-instantons. In the large N limit multi-instanton effects in $\mathcal{N}=4$ SYM can be explicitly computed and compared with AdS amplitudes involving certain D-instanton induced vertices in the type IIB effective action. Such vertices appear in the α' expansion, *i.e.* the low energy derivative expansion, of type IIB supergravity and their structure is determined by supersymmetry and S-duality constraints. For a large class of processes a striking agreement between supergravity multi-particle amplitudes involving the leading D-instanton vertices and multi-instanton contributions to the dual Yang–Mills correlation functions was found in [35], generalising the results of [34, 36]. The correspondence has been further extended to include higher order D-instanton terms in [37]. What makes the result of the comparison remarkable is that perfect agreement is found in the functional form of correlation functions which depend non-trivially on the

coupling.

The non-perturbative results just described refer to a class of four- and higher-point correlation functions of protected operators. All the cases analysed in the past involve 1/2 BPS operators, which are dual to supergravity states and their Kaluza–Klein excited modes. In view of the success in the comparison with D-instanton effects at the level of supergravity on the one hand and of the recent progress in matching the gauge and string theory spectra at the level of massive string excitations on the other, it is interesting to study instanton contributions to correlation functions of operators dual to massive string modes. In the present paper we shall consider two-point functions of gauge-invariant scalar operators of bare dimension $\Delta_0 \leq 5$. These include BPS operators dual to supergravity fields, multi-trace operators dual to multi-particle bound states as well as single-trace operators in long multiplets dual to massive excitations of type IIB string theory.

As is well known, two-point functions in a conformal field theory like $\mathcal{N}=4$ supersymmetric Yang–Mills encode the information about scaling dimensions. For primary operators $\mathcal{O}(x)$ conformal invariance determines the form of two-point functions to be

$$\langle \bar{\mathcal{O}}(x) \mathcal{O}(y) \rangle = \frac{c}{(x-y)^{2\Delta}}, \quad (2)$$

where c is a constant and the scaling dimension is $\Delta = \Delta_0 + \gamma$, with γ the anomalous part which has an expansion at weak coupling. By computing (2) at the instanton level one can extract the corresponding contribution to γ . In general the anomalous dimensions have an expansion of the form

$$\gamma(g_{\text{YM}}, N) = \sum_{n=1}^{\infty} \gamma_n^{\text{pert}}(N) g_{\text{YM}}^{2n} + \sum_{K>0} \sum_{m=0}^{\infty} \left[\gamma_m^{(K)}(N) g_{\text{YM}}^{2m} e^{2\pi i \tau K} + \text{c.c.} \right], \quad (3)$$

where $\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g_{\text{YM}}^2}$ and anti-instantons give the complex conjugate of the instanton contributions, since the anomalous dimensions are real quantities.

The $\mathcal{N}=4$ theory is believed to possess a $\text{SL}(2, \mathbb{Z})$ S-duality symmetry under which the complexified coupling τ transforms projectively, $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ (where $ad-bc=1$ with integer a, b, c, d). In the dual type IIB string theory D-instantons are instrumental in implementing the constraints imposed by S-duality, for instance on the form of the low energy effective action. On the field theory side S-duality requires the invariance of the spectrum of scaling dimensions and instantons are expected to play a crucial rôle in the implementation of this symmetry. The invariance of the spectrum under $\text{SL}(2, \mathbb{Z})$ transformations suggests that single anomalous dimensions should be functions of τ and $\bar{\tau}$,

$$\Delta = \Delta(\tau, \bar{\tau}; N) \equiv \Delta_0 + \gamma(\tau, \bar{\tau}; N), \quad (4)$$

so that in general a contribution from the non-perturbative part in (3) is expected.

In order to compute instanton corrections to anomalous dimensions in the following we shall study two-point functions of gauge-invariant scalar operators of bare dimension $\Delta_0 = 2, 3, 4$ and 5. The calculations will be performed in the semiclassical approximation and in the one-instanton sector of the $\text{SU}(N)$ $\mathcal{N}=4$ supersymmetric Yang–Mills theory. This means that expectation values reduce to finite dimensional integrals over the moduli

space parametrised by the bosonic and fermionic collective coordinates which arise in the instanton background. The general aspects of these calculations are reviewed in section 2 and the case of two-point correlators is discussed in section 4.1. The methods developed in this paper can be applied to non Lorentz-scalar operators as well. In this case additional quantum numbers need to be taken into account in the construction of independent sets of operators, but the calculation of instanton corrections to two-point correlation functions does not present any new complications.

The quantum numbers that characterise scalar operators are, besides the dimension Δ_0 , the Dynkin labels, $[a, b, c]$, determining their transformation under the $SU(4)$ R-symmetry. The parameters $(\Delta_0; [a, b, c])$ identify sectors in the theory and as discussed in section 4.2 in order to extract the anomalous dimensions one has to compute all the two-point functions to resolve the mixing among operators in any given sector. One of the outcomes of our analysis is that the problem of operator mixing is in general more complicated at the instanton level than in perturbation theory: in sectors where instantons contribute we find that mixing occurs among all the operators at the same leading order in the coupling. We shall not compute all the entries in the mixing matrices for the sectors we shall examine and thus it will not be possible to pin down the actual values of the instanton induced anomalous dimensions. However it will be possible, analysing the moduli space integrations, to unambiguously identify the sectors in which the scaling dimensions receive instanton corrections.

Rather surprisingly the calculations of section 5 show that the majority of the scalar operators of dimension $\Delta_0 \leq 5$ do not get instanton corrections. This is an unexpected result in view of the above considerations on S-duality: many of the operators that we examine are corrected in perturbation theory, but not at the instanton level. A similar behaviour had been observed in [4, 38] for operators in the Konishi multiplet. The results presented here might suggest that this is a more general feature of the theory, however in the concluding discussion we shall argue that this is probably not the case and that the non-renormalisation properties we find are likely to be specific to operators of small dimension. At $\Delta_0 = 2$ there are two operators and all their two-point functions vanish in the one-instanton sector. This was a known result since one of the operators belongs to a protected $1/2$ BPS multiplet and the other is a component of the Konishi multiplet, whose anomalous dimension was shown not to receive instanton corrections in [4, 38]. Among the operators of dimension 3 there is only one whose structure allows instanton contributions, it is the unique operator $\mathcal{O}_{3,6}^i$ in the **6** of $SU(4)$. However, although this is hard to see from the two-point functions, we shall give another argument, based on the OPE analysis of the four-point function computed in appendix C, showing that $\mathcal{O}_{3,6}^i$ has no instanton induced anomalous dimension. In all the other sectors at $\Delta_0 = 3$, corresponding to the representations **10**, $\overline{\mathbf{10}}$ and **50**, there are no instanton corrections. At $\Delta_0 = 4$ we find the first non-vanishing instanton contributions. For operators in the $SU(4)$ singlet we find non-vanishing two-point functions. Even without computing the actual values of the anomalous dimensions, the results of section 5.3.1 show that operators in this sector have scaling dimensions which are corrected by instantons. Operators in all the remaining sectors at $\Delta_0 = 4$, *i.e.* in the $SU(4)$ representations **15**, **20'**, **45**, $\overline{\mathbf{45}}$, **84** and **105**, are not renormalised by instantons. Similarly among the operators of bare dimension $\Delta_0 = 5$ only those transforming in the **6** have non-vanishing two-point functions in the one-instanton

background. These are more complicated to compute because their evaluation at leading order in the coupling requires the inclusion of the leading quantum fluctuations around the instanton configuration. Explicit examples of calculations of such effects will be given in section 5.4.1. Again, in all the other $SU(4)$ sectors at $\Delta_0 = 5$ (**10**, **$\overline{10}$** , **50**, **64**, **70**, **$\overline{70}$** , **126**, **$\overline{126}$** , **196** and **300**) no instanton corrections to two-point functions are found. For both the singlet at $\Delta_0 = 4$ and the **6** at $\Delta_0 = 5$ the OPE analysis of the four-point function computed in appendix C confirms the presence of non-perturbative corrections to the scaling dimensions.

In the calculation of anomalous dimensions important simplifications arise from the use of the constraints imposed by the $PSU(2,2|4)$ global symmetry. First of all supersymmetry implies that all the operators in a multiplet must have the same anomalous dimension. Therefore the above results for operators with $\Delta_0 \leq 5$ imply the absence of instanton corrections to a much larger set of operators: all the components of the multiplets for which a representative is found to be non-renormalised share the same (vanishing) anomalous dimension. Taking into account this fact and the structure of the $PSU(2,2|4)$ multiplets, which can be determined systematically using the method of [39], the problem of extracting the values of the anomalous dimension for operators that receive corrections can be significantly simplified. When a direct analysis is complicated it is usually possible to identify superconformal descendants which can be more easily studied. Then superconformal symmetry implies that the result for the anomalous dimension extends to the original operators. An explicit example of this is represented by the case of the Konishi multiplet: the absence of instanton corrections to the superconformal primary (as well as to the whole multiplet) is obtained as consequence of the non-renormalisation of a suitably chosen descendant. The same idea can be used to simplify the resolution of the operator-mixing when instanton corrections are present.

As already observed the above results for $\Delta_0 \leq 5$ probably do not reflect a general property of the theory, but rather are characteristic of small dimension sectors. However some non-renormalisation properties can be generalised to arbitrary Δ_0 . This is in particular the case for the absence of instanton corrections in the $SU(2)$ subsector identified in [29] consisting of scalar operators transforming in the representation $[a, b, a]$ and with bare dimension $\Delta_0 = 2a + b$. In [29, 41] it was shown that the action of the dilation operator on such operators is particularly simple. We shall argue that operators of this type do not receive instanton corrections. Superconformal symmetry implies that this result is valid for all the components of multiplets containing operators of this type.

The paper is organised as follows. In section 2 we review general aspects of instanton calculus in $\mathcal{N}=4$ SYM. Bosonic and fermionic collective coordinates in the one-instanton sector are described together with the integration measure on the moduli space used in the semiclassical calculations. The structure of the $\mathcal{N}=4$ instanton supermultiplet, *i.e.* of the instanton solution for the elementary fields, is described in section 3. Generalities on the calculation of instanton induced two-point functions and their relation with corrections to the anomalous dimensions are discussed in section 4. Section 5 presents the calculation of one-instanton contributions to two-point functions of operators with $\Delta_0 \leq 5$ and contains the main results of the paper. The final section contains a discussion of the results and some comments on the possibility of realising the action of the dilation operator on the instanton moduli space. Appendix A summarises our notation and in appendix B a brief

summary of the ADHM description of the one-instanton sector of $\mathcal{N}=4$ SYM is provided. In appendix C we present the calculation of a four-point function, which allows to resolve ambiguities left from the analysis of two-point functions in sections 5.2.1 and 5.4.1.

2 One-instanton contributions to correlation functions in $\mathcal{N}=4$ SYM

In this section we present general aspects of the computation of instanton effects in semiclassical approximation in the $\mathcal{N}=4$ supersymmetric Yang–Mills theory.

The general instanton configuration can be described using the ADHM formalism [42]. The original construction of self-dual configurations in pure Yang–Mills theory with arbitrary classical gauge group has been generalised to the case of theories with different field content and in particular to supersymmetric theories. Comprehensive reviews of instanton calculus in supersymmetric gauge theories can be found in [43,44]. In particular [44] contains a detailed description of multi-instantons in the $\mathcal{N}=4$ SYM theory. In the following we shall focus on the one-instanton sector of the $\mathcal{N}=4$ theory with $SU(N)$ gauge group. Many of the results of the present paper however remain valid in sectors with arbitrary instanton number, K . The extension to orthogonal and symplectic groups also does not present particular problems from the point of view of the instanton calculations. Appendix B contains a brief summary of the ADHM description of the one-instanton sector of the $SU(N)$ $\mathcal{N}=4$ SYM theory.

We begin by discussing instanton contributions to generic correlation functions of gauge invariant composite operators in semiclassical approximation. The specific case of two-point functions of such operators, which is relevant for the computation of instanton-induced anomalous dimensions will be considered in section 4.

At the semiclassical level expectation values are evaluated using a saddle point approximation around the instanton configuration. For a correlation function of generic local operators \mathcal{O}_i the path integral reduces to a finite dimensional integration over the moduli space of the instanton, *i.e.* an integration over the unfixed parameters (moduli or collective coordinates) of the generic instanton configuration as described by the ADHM construction

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \int d\mu_{\text{inst}}(m_b, m_f) e^{-S_{\text{inst}}} \hat{\mathcal{O}}_1(x_1; m_b, m_f) \dots \hat{\mathcal{O}}_n(x_n; m_b, m_f). \quad (5)$$

In (5) we have denoted the bosonic and fermionic collective coordinates by m_b and m_f respectively and by $d\mu_{\text{inst}}(m_b, m_f)$ the corresponding integration measure; S_{inst} is the classical action evaluated on the instanton solution and $\hat{\mathcal{O}}_r$, $r = 1, \dots, n$, denotes the classical expression for the operator \mathcal{O}_r computed in the instanton background.

In the one-instanton sector and with $SU(N)$ gauge group there are $4N$ bosonic collective coordinates entering into the integration measure in (5). With a particular choice of parametrisation these bosonic moduli can be identified with the size, ρ , and position, x_0 , of the instanton as well as its global gauge orientation. The latter can in turn be described by three angles identifying an $SU(2)$ instanton and $4N$ additional constrained variables, $w_{u\dot{\alpha}}$ and $\bar{w}^{\dot{\alpha}u}$ (where u is a colour index), in the coset $SU(N)/(SU(N-2) \times U(1))$ describing the embedding of the $SU(2)$ configuration into $SU(N)$. In the $\mathcal{N}=4$ theory

there are $8N$ fermionic collective coordinates as well in the one-instanton sector. These correspond to zero modes of the Dirac operator in the background of an instanton. They comprise the 16 moduli associated with Poincaré and special supersymmetries broken by the instanton and denoted respectively by η_α^A and $\bar{\xi}^{\dot{\alpha}A}$ (where A is an index in the fundamental of the $SU(4)$ R-symmetry group) and $8N$ additional parameters, ν_u^A and $\bar{\nu}^{Au}$, which can be considered as the fermionic superpartners of the gauge orientation parameters. The fermion modes ν_u^A and $\bar{\nu}^{Au}$ satisfy the constraints

$$\bar{w}^{\dot{\alpha}u}\nu_u^A = 0, \quad \bar{\nu}^{Au}w_{u\dot{\alpha}} = 0, \quad (6)$$

which effectively reduce the number of independent variables of this type to $8(N-2)$.

In the computation of correlation functions of gauge-invariant operators as the ones we shall be interested in the moduli space integration measure in (5) simplifies. The only non-trivial bosonic integrals are over ρ and x_0 . Moreover in the case of the $\mathcal{N}=4$ theory of the total set of $8N$ fermionic moduli dictated by the index theorem, only the η_α^A and $\bar{\xi}^{\dot{\alpha}A}$ moduli associated with broken supersymmetries correspond to true zero-modes when the interactions are taken into account. The instanton action acquires a non-trivial dependence on the other modes, ν_u^A and $\bar{\nu}^{Au}$, which means that the integration measure has an explicit dependence on these modes. The moduli space integration measure for the $\mathcal{N}=4$ supersymmetric Yang–Mills theory in the generic K instanton sector was constructed in [35]. In the one-instanton sector the gauge-invariant measure takes the form

$$\begin{aligned} & \int d\mu_{\text{phys}} e^{-S_{\text{inst}}} \\ &= \frac{\pi^{-4N} g_{\text{YM}}^{4N} e^{2\pi i\tau}}{(N-1)!(N-2)!} \int d\rho d^4x_0 \prod_{A=1}^4 d^2\eta^A d^2\bar{\xi}^A d^{N-2}\nu^A d^{N-2}\bar{\nu}^A \rho^{4N-13} e^{-S_{4F}}, \end{aligned} \quad (7)$$

where the instanton action is [45]

$$S_{\text{inst}} = -2\pi i\tau + S_{4F} = -2\pi i\tau + \frac{\pi^2}{2g_{\text{YM}}^2 \rho^2} \varepsilon_{ABCD} \mathcal{F}^{AB} \mathcal{F}^{CD} \quad (8)$$

with

$$\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}, \quad \mathcal{F}^{AB} = \frac{1}{2\sqrt{2}} (\bar{\nu}^{Au} \nu_u^B - \bar{\nu}^{Bu} \nu_u^A). \quad (9)$$

In (7) we have omitted an overall numerical constant, independent of g_{YM} and N , that will be reinstated in the final expression. Following [36] the integration measure can be written in the form

$$\begin{aligned} & \frac{\pi^{-4N} g_{\text{YM}}^{4N} e^{2\pi i\tau}}{(N-1)!(N-2)!} \int d\rho d^4x_0 d^6\chi \prod_{A=1}^4 d^2\eta^A d^2\bar{\xi}^A d^{N-2}\nu^A d^{N-2}\bar{\nu}^A \\ & \rho^{4N-7} \exp \left[-2\rho^2 \chi^i \chi^i + \frac{4\pi i}{g_{\text{YM}}} \chi_{AB} \mathcal{F}^{AB} \right], \end{aligned} \quad (10)$$

where auxiliary bosonic variables, χ^i , $i = 1, \dots, 6$, have been introduced to rewrite the exponential of the quartic fermionic action as a gaussian integral. In (10)

$$\chi_{AB} = \frac{1}{\sqrt{8}} \Sigma_{AB}^i \chi^i, \quad (11)$$

where the symbols Σ_{AB}^i denote Clebsch-Gordan coefficients projecting the product of two $\overline{4}$'s of $SU(4)$ onto the $\mathbf{6}$ and are defined in appendix A.

Once the measure on the instanton moduli space is written in the form (10) the integration over ν_u^A and $\bar{\nu}^{Au}$ is gaussian and can be immediately performed. However in general in correlation functions of gauge-invariant operators there is also a non-trivial dependence on these variables coming from the expressions for the operators in the instanton background. It is thus convenient to construct a generating function as in [37], which allows to deal easily with the otherwise complicated combinatorics associated with the fermionic integrations over ν_u^A and $\bar{\nu}^{Au}$. We introduce sources, $\bar{\vartheta}_A^u$ and ϑ_{Au} , coupled to ν_u^A and $\bar{\nu}^{Au}$ and define

$$Z[\vartheta, \bar{\vartheta}] = \frac{\pi^{-4N} g_{\text{YM}}^{4N} e^{2\pi i \tau}}{(N-1)!(N-2)!} \int d\rho d^4 x_0 d^6 \chi \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A d^{N-2} \bar{\nu}^A d^{N-2} \nu^A \\ \rho^{4N-7} \exp \left[-2\rho^2 \chi^i \chi^i + \frac{\sqrt{8}\pi i}{g_{\text{YM}}} \bar{\nu}^{Au} \chi_{AB} \nu_u^B + \bar{\vartheta}_A^u \nu_u^A + \vartheta_{Au} \bar{\nu}^{Au} \right]. \quad (12)$$

Performing the gaussian $\bar{\nu}$ and ν integrals and introducing polar coordinates,

$$\chi^i \rightarrow (r, \Omega), \quad \sum_{i=1}^6 (\chi^i)^2 = r^2, \quad (13)$$

$Z[\vartheta, \bar{\vartheta}]$ can be put in the form

$$Z[\vartheta, \bar{\vartheta}] = \frac{2^{-29} \pi^{-13} g_{\text{YM}}^8 e^{2\pi i \tau}}{(N-1)!(N-2)!} \int d\rho d^4 x_0 d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \rho^{4N-7} \\ \int_0^\infty dr r^{4N-3} e^{-2\rho^2 r^2} \mathcal{Z}(\vartheta, \bar{\vartheta}; \Omega, r), \quad (14)$$

where all the numerical coefficients have been reinstated. In (14) we have introduced the density

$$\mathcal{Z}(\vartheta, \bar{\vartheta}; \Omega, r) = \exp \left[-\frac{i g_{\text{YM}}}{\pi r} \bar{\vartheta}_A^u \Omega^{AB} \vartheta_{Bu} \right], \quad (15)$$

where the symplectic form Ω^{AB} is given by

$$\Omega^{AB} = \frac{1}{\sqrt{8}} \bar{\Sigma}_i^{AB} \Omega^i, \quad \sum_{i=1}^6 (\Omega^i)^2 = 1 \quad (16)$$

and the symbols $\bar{\Sigma}_i^{AB}$ are defined in appendix A.

In conclusion the semiclassical approximation to a correlation function (5) is computed as

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{2^{-29} \pi^{-13} g_{\text{YM}}^8 e^{2\pi i \tau}}{(N-1)!(N-2)!} \int d\rho d^4 x_0 d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \rho^{4N-7} \\ \int_0^\infty dr r^{4N-3} e^{-2\rho^2 r^2} \left[\mathcal{Z}(\vartheta, \bar{\vartheta}; \Omega, r) \hat{\mathcal{O}}_1(x_1; \rho, x_0; \eta, \bar{\xi}, \nu(\Omega), \bar{\nu}(\Omega)) \right. \\ \left. \dots \hat{\mathcal{O}}_n(x_n; \rho, x_0; \eta, \bar{\xi}, \nu(\Omega), \bar{\nu}(\Omega)) \right] \Big|_{\vartheta=\bar{\vartheta}=0}, \quad (17)$$

where the ν_u^A and $\bar{\nu}^{Au}$ variables in each $\hat{\mathcal{O}}_r$ are understood to be rewritten in terms of derivatives with respect to the sources, $\bar{\vartheta}_A^u$ and ϑ_{Au} , before setting the latter to zero. As discussed in [37] the use of the generating function to express a correlator as in (17) allows in particular to easily determine the dependence on the parameters g_{YM} and N .

The variables ν_u^A and $\bar{\nu}^{Au}$ enter in the expressions of gauge-invariant composite operators in the instanton background always in colour singlet bilinears. They arise in pairs either in a combination which is in the **6** of the SU(4) R-symmetry

$$(\bar{\nu}^A \nu^B)_{\mathbf{6}} \equiv \bar{\nu}^{u[A} \nu_u^{B]} = (\bar{\nu}^{Au} \nu_u^B - \bar{\nu}^{Bu} \nu_u^A), \quad (18)$$

or in the **10** of SU(4)

$$(\bar{\nu}^A \nu^B)_{\mathbf{10}} \equiv \bar{\nu}^{u(A} \nu_u^{B)} = (\bar{\nu}^{Au} \nu_u^B + \bar{\nu}^{Bu} \nu_u^A). \quad (19)$$

Using the generating function defined in (14) we can determine the dependence of a generic correlator on the parameters g_{YM} and N . In the one-instanton sector the N -dependence can be computed exactly. In particular in the large- N limit from (17) we get

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle &\sim \alpha(p, q; N) g_{\text{YM}}^{8+p+q} e^{2\pi i \tau} \int d\rho d^4 x_0 d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \rho^{p+q-5} \\ &\quad \hat{\mathcal{O}}_1(x_1; \rho, x_0; \eta, \bar{\xi}, \bar{\nu}\nu(\Omega)) \dots \hat{\mathcal{O}}_n(x_n; \rho, x_0; \eta, \bar{\xi}, \bar{\nu}\nu(\Omega)), \end{aligned} \quad (20)$$

with

$$\begin{aligned} \alpha(p, q; N) &= \frac{2^{-2N+\frac{1}{2}(p+q)} \pi^{-(p+q)} \Gamma\left(2N-1-\frac{1}{2}(p+q)\right)}{(N-1)!(N-2)!} \left(N^{p+\frac{q}{2}} + O(N^{p+\frac{q}{2}-1})\right) \\ &\sim N^{\frac{1}{2}(p+1)} (1 + O(1/N)), \end{aligned} \quad (21)$$

where p and q denote respectively the number of $(\bar{\nu}\nu)_{\mathbf{6}}$ and $(\bar{\nu}\nu)_{\mathbf{10}}$ bilinears entering the integrand in (20).

As will be discussed explicitly in the case of scalar operators of bare dimension 5, in general the computation of two-point functions in the instanton background at the first non-trivial order in the coupling, g_{YM} , requires to take into account the effect of the leading quantum fluctuations around the classical configuration. To compute leading order effects we need to include contributions in which, instead of replacing all the fields with their background value in the presence of an instanton, pairs of fields are contracted via a propagator. The general construction of the scalar propagator in the instanton background in terms of ADHM variables was given in [46], an explicit expression for the scalar propagator in the adjoint representation of the SU(N) gauge group was presented in [37]. In general the propagators for vectors and spinors are also needed. These propagators can be deduced from the scalar propagator as discussed in [47]. The Green function for the adjoint scalars which will be relevant for some of the calculations in section 5.4 is given in appendix B.

3 The $\mathcal{N}=4$ instanton supermultiplet

As discussed in the previous section, in order to evaluate instanton induced correlation functions we need to integrate the classical profiles ¹ of the relevant composite operators over the instanton moduli space. In preparation for such computations in this section we shall discuss the structure of the instanton solution for the elementary fields in the $\mathcal{N}=4$ supermultiplet. We are interested in the dependence on the collective coordinates and of particular relevance will be the way the fermionic modes enter into the expressions for the various fields.

The $\mathcal{N}=4$ multiplet consists of six real scalars, φ^{AB} , four Weyl spinors, λ_α^A , and a vector, A_μ . Our notation is summarised in appendix A. The field equations in the $\mathcal{N}=4$ SYM theory take the form

$$\begin{aligned} \mathcal{D}_\mu F^{\mu\nu} + i\{\lambda^{\alpha A}\sigma_{\alpha\dot{\alpha}}^\nu, \bar{\lambda}_{\dot{\alpha}A}\} + \frac{1}{2}[\bar{\varphi}_{AB}, \mathcal{D}^\nu \varphi^{AB}] &= 0 \\ \mathcal{D}^2 \varphi^{AB} + \sqrt{2}\{\lambda^{\alpha A}, \lambda_\alpha^B\} + \frac{1}{\sqrt{2}}\varepsilon^{ABCD}\{\bar{\lambda}_{\dot{\alpha}C}, \bar{\lambda}_{\dot{\alpha}D}\} - \frac{1}{2}[\bar{\varphi}_{CD}, [\varphi^{AB}, \varphi^{CD}]] &= 0 \\ \bar{\mathcal{D}}_{\dot{\alpha}\alpha}\lambda^{\alpha A} + i\sqrt{2}[\varphi^{AB}, \bar{\lambda}_{\dot{\alpha}B}] &= 0 \\ \mathcal{D}_{\alpha\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}A} - i\sqrt{2}[\bar{\varphi}_{AB}, \lambda_\alpha^B] &= 0. \end{aligned} \tag{22}$$

A solution to these equations is given by

$$A_\mu = A_\mu^I, \quad \varphi^{AB} = \lambda_\alpha^A = \bar{\lambda}_{\dot{\alpha}A} = 0, \tag{23}$$

where A_μ^I is the standard instanton solution of $SU(N)$ pure Yang–Mills theory. However the Dirac operator has zero modes in the background of this solution, *i.e.* the equation $\bar{\mathcal{D}}_{\dot{\alpha}\alpha}\lambda^{\alpha A} = 0$ has non-trivial solutions when the covariant derivative is evaluated in the background of the instanton. This is the origin of the fermionic zero-mode integrations in the semiclassical expression for correlation functions (17). Because of these fermionic integrations using the solution (23) to compute the classical profiles of the operators $\hat{\mathcal{O}}_i$ in (17) is not sufficient and we need to include the zero-mode dependence in the operators. We must thus solve the equations (22) iteratively in order to determine the complete dependence on the fermion zero modes in the multiplet of elementary fields [48]. In the following we shall use the notation $\Phi^{(n)}$ to denote a term in the solution for the field Φ containing n fermion zero modes.

Notice that in the simple case of $SU(2)$ gauge group there are only 16 fermionic modes in a one-instanton background, the ones associated with broken superconformal symmetries. In this case it is possible to determine completely the zero mode dependence in the $\mathcal{N}=4$ supermultiplet acting with the broken supersymmetries on (23). Substituting $A_\mu^{(0)} \equiv A_\mu^I$ in the supersymmetry transformation of λ_α^A gives a configuration, $\lambda_\alpha^{(1)A}$, which is linear in the fermion modes η_α^A and $\bar{\xi}^{\dot{\alpha}A}$ and solves the corresponding field equation. Then plugging $\lambda^{(1)A}$ into the variation of φ^{AB} generates a solution for the scalar which

¹In the following we shall use the word “profile” in a slightly loose sense to indicate the expression for a composite operator computed in an instanton background including the dependence on fermion zero-modes.

is quadratic in the fermion modes, $\varphi^{(2)AB}$. Iteration of this procedure gives rise to a solution $\bar{\lambda}_{\dot{\alpha}A}^{(3)}$ and then to a contribution to A_μ which corrects the original solution with the addition of a term quartic in the fermion modes, $A_\mu^{(4)}$. In this way one can construct the complete dependence on the fermion zero modes in the $\mathcal{N}=4$ supermultiplet in the case of SU(2) gauge group. The iteration continues until the number of zero modes in the fields exceeds 16, at which point further variations do not produce new independent terms in the solutions. This procedure can be implemented in an efficient way using a superspace formalism. A general discussion in the case of $\mathcal{N}=1$ supersymmetric Yang–Mills theory can be found in [49].

In the case of SU(N) gauge group the situation is more complicated and the above procedure cannot be utilised. As already discussed in this case there are the additional fermion zero modes ν_u^A and $\bar{\nu}^{Au}$ which are not associated with symmetries broken by the bosonic instanton solution. The dependence on these modes, which is crucial in computing Green functions in semiclassical approximation, cannot be obtained using symmetry arguments. Therefore we need to explicitly construct the zero-mode dependence by solving the field equations.

Starting with the solution $A_\mu^{(0)}$ in (23), we solve the equations (22) iteratively to generate solutions for all the fields in the multiplet with an increasing number of fermionic zero-modes. The first few steps in this construction have been carried out in [48].

The term linear in the fermion modes in the solution for the spinor, $\lambda_\alpha^{(1)A}$, is determined by the equation

$$\bar{\mathcal{D}}^{(0)\dot{\alpha}\alpha}\lambda_\alpha^{(1)A} = 0, \quad (24)$$

where the covariant derivative $\bar{\mathcal{D}}^{(0)\dot{\alpha}\alpha}$ contains $A_\mu^{(0)}$. The subsequent steps give rise to $\varphi^{(2)AB}$ and $\bar{\lambda}_{\dot{\alpha}A}^{(3)}$, which are obtained solving respectively

$$\mathcal{D}^{(0)2}\varphi^{(2)AB} + \sqrt{2}\{\lambda^{(1)\alpha A}, \lambda_\alpha^{(1)B}\} = 0 \quad (25)$$

and

$$\mathcal{D}_{\alpha\dot{\alpha}}^{(0)}\bar{\lambda}_A^{(3)\dot{\alpha}} - i\sqrt{2}[\bar{\varphi}_{AB}^{(2)}, \lambda_\alpha^{(1)B}] = 0. \quad (26)$$

Further iteration gives rise to corrections to the above lowest order solution in which each field has the minimal number of fermion modes, $\{A_\mu^{(0)}, \lambda_\alpha^{(1)A}, \varphi^{(2)AB}, \bar{\lambda}_A^{(3)\dot{\alpha}}\}$. The following steps generate the terms $A_\mu^{(4)}$, $\lambda_\alpha^{(5)A}$ and $\varphi^{(6)AB}$ which solve respectively

$$\mathcal{D}^{(0)2}A_\mu^{(4)} - \mathcal{D}_\mu^{(0)}\mathcal{D}_\nu^{(0)}A^{(4)\nu} + 2[F_{\mu\nu}^{(0)}, A^{(4)\nu}] - i\{\bar{\lambda}_{\dot{\alpha}A}^{(3)}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}, \lambda_\alpha^{(1)A}\} = 0, \quad (27)$$

$$\bar{\mathcal{D}}_{\dot{\alpha}\alpha}^{(0)}\lambda^{(5)\alpha A} + \bar{\mathcal{D}}_{\dot{\alpha}\alpha}^{(4)}\lambda^{(1)\alpha A} + i\sqrt{2}[\varphi^{(2)AB}, \bar{\lambda}_{\dot{\alpha}B}^{(3)}] = 0 \quad (28)$$

and

$$\begin{aligned} &\mathcal{D}^{(0)2}\varphi^{(6)AB} + \mathcal{D}^{(4)2}\varphi^{(2)AB} + \sqrt{2}\{\lambda^{(1)\alpha A}, \lambda_\alpha^{(5)B}\} + \sqrt{2}\{\lambda^{(5)\alpha A}, \lambda_\alpha^{(1)B}\} \\ &+ \frac{1}{\sqrt{2}}\varepsilon^{ABCD}\{\bar{\lambda}_{\dot{\alpha}C}^{(3)}, \bar{\lambda}_D^{(3)\dot{\alpha}}\} - \frac{1}{2}[\bar{\varphi}_{CD}^{(2)}, [\varphi^{(2)AB}, \varphi^{(2)CD}]] = 0. \end{aligned} \quad (29)$$

For generic operators in the $\mathcal{N}=4$ supersymmetric Yang–Mills theory these new terms contribute and are needed to compute correlation functions *at leading order in g_{YM}* . In the

special case of 1/2 BPS operators, as those in the supercurrent multiplet, the AdS/CFT correspondence suggests that only terms with the minimal number of superconformal modes are allowed [37]. Operators of this type are dual to the supergravity multiplet and its Kaluza–Klein excitations in the type IIB string theory in $\text{AdS}_5 \times S^5$ and the AdS/CFT correspondence combined with knowledge of the structure of the low energy effective action for the type IIB “massless” fields puts restrictions on the set of correlation functions which can receive instanton contributions. Such constraints restrict the maximal number of superconformal modes that each operator can saturate. The same constraints do not apply to the unprotected operators we shall consider in the following and thus higher order terms will be needed as well.

The procedure outlined here can in principle be employed to determine the exact zero-mode structure of the solution. The actual implementation of this construction becomes soon very involved as one gets to higher order terms. The general solution takes the form

$$\begin{aligned} A_\mu &= \sum_{\substack{n=0 \\ 4n \leq 8N}} A_\mu^{(4n)}, & \varphi^{AB} &= \sum_{\substack{n=0 \\ 4n+2 \leq 8N}} \varphi^{(4n+2)AB} \\ \lambda_\alpha^A &= \sum_{\substack{n=0 \\ 4n+1 \leq 8N}} \lambda_\alpha^{(4n+1)A}, & \bar{\lambda}_{\dot{\alpha}A} &= \sum_{\substack{n=0 \\ 4n+3 \leq 8N}} \bar{\lambda}_{\dot{\alpha}A}^{(4n+3)}, \end{aligned} \quad (30)$$

where it is also understood that in each field the number of superconformal modes does not exceed 16 and the remaining modes are of ν_u^A and $\bar{\nu}^{Au}$ type.

In computing the expressions for gauge invariant composite operators we shall make use of the ADHM description in which the elementary fields are written as $[N+2] \times [N+2]$ matrices as discussed in appendix B. In the same appendix the leading order terms in the solution for A_μ , λ_α^A and φ^{AB} , which will be used in some of the examples presented in section 5, are given explicitly.

Here we shall not discuss the details of the solution of the iterative equations, but instead we shall only analyse the $\text{SU}(4)$ structure, which will suffice for the study of two-point functions to be carried out in later sections. The scalar fields in the $\mathcal{N}=4$ multiplet, $\varphi^i \sim \varphi^{AB}$, transform in the representation **6** (with Dynkin labels $[0, 1, 0]$) of the $\text{SU}(4)$ R-symmetry group, the fermions, λ_α^A and $\bar{\lambda}_{\dot{\alpha}A}$, transform respectively in the **4** ($[1, 0, 0]$) and $\bar{\mathbf{4}}$ ($[0, 0, 1]$) and the vector, A_μ , (as well as its field strength and the covariant derivatives) is a singlet. The $\text{SU}(4)$ structure of the combination of fermion zero modes in the various terms in the iterative solution discussed above can be determined without solving the equations explicitly. All the fermion zero modes, both the superconformal ones, η_α^A and $\bar{\xi}^{\dot{\alpha}A}$, and the modes of type ν_u^A and $\bar{\nu}^{Au}$, transform in the **4** of $\text{SU}(4)$. We shall denote a generic fermion mode by m_f^A . Inspecting the iterative equations that determine the instanton multiplet we can deduce in which $\text{SU}(4)$ combinations the fermion modes enter each term. The starting point is the classical instanton, $A_\mu^{(0)}$, which has no fermions. The first term in λ_α^A is linear in the fermion modes

$$\lambda_\alpha^{(1)A} \sim m_f^A. \quad (31)$$

For the term $\varphi^{(2)AB}$ in the scalar solution we find

$$\varphi^{(2)AB} \sim m_f^A m_f^B, \quad (32)$$

i.e. the two fermion modes are antisymmetrised in order to form a combination in the **6**. The schematic notation of (32) indicates that the $[N+2] \times [N+2]$ matrix $\varphi^{(2)AB}$ has entries which involve one mode of flavour A and one of flavour B in all the possible combinations

$$\eta_\alpha^{[A}\eta_\beta^{B]}, \quad \eta_\alpha^{[A}\bar{\xi}^{\dot{\alpha}B]}, \quad \bar{\xi}^{\dot{\alpha}[A}\bar{\xi}^{\dot{\beta}B]}, \quad \eta_\alpha^{[A}\bar{\nu}^{B]v}, \quad \nu_u^{[A}\eta_\alpha^{B]}, \quad \bar{\xi}^{\dot{\alpha}[A}\bar{\nu}^{B]v}, \quad \nu_u^{[A}\bar{\xi}^{\dot{\beta}B]}, \quad \nu_u^{[A}\bar{\nu}^{B]v}. \quad (33)$$

The $\bar{\mathbf{4}}$ spinor $\bar{\lambda}_A^{(3)\dot{\alpha}}$ contains fermion modes in the combination

$$\bar{\lambda}_A^{(3)\dot{\alpha}} \sim \varepsilon_{ABCD} m_f^B m_f^C m_f^D, \quad (34)$$

so that the component $\lambda_A^{(3)}$ has three fermion modes, one of each of the flavours apart from A . Proceeding in the multiplet we find the quartic term in the solution for the vector, $A_\mu^{(4)}$, which contains one fermion mode of each flavour in a singlet combination

$$A_\mu^{(4)} \sim \varepsilon_{ABCD} m_f^A m_f^B m_f^C m_f^D. \quad (35)$$

The following term is $\lambda_\alpha^{(5)A}$, which has flavour structure

$$\lambda_\alpha^{(5)A} \sim \varepsilon_{A'B'C'D'} m_f^A m_f^{A'} m_f^{B'} m_f^{C'} m_f^{D'}, \quad (36)$$

i.e. it involves a mode of flavour A plus one of each flavour. Then we find $\varphi^{(6)AB}$ that contains an antisymmetric combination of a mode of flavour A and one of flavour B plus one mode of each flavour

$$\varphi^{(6)AB} \sim \varepsilon_{A'B'C'D'} m_f^{[A} m_f^{B]} m_f^{A'} m_f^{B'} m_f^{C'} m_f^{D'}. \quad (37)$$

As observed after equation (32) the previous expressions are symbolic and the products of m_f 's in (34)-(37) correspond to different combinations of the modes η_α^A , $\bar{\xi}^{\dot{\alpha}A}$, ν_u^A and $\bar{\nu}^{Au}$ in the various entries of the ADHM matrices for each field.

The $SU(4)$ structure of the combinations of fermionic modes entering into the higher order terms in the solution can be determined analogously. As already mentioned the knowledge of the flavour structure of the solution described here will be enough to obtain some interesting results concerning the anomalous dimensions of composite operators without actually solving the field equations.

4 Instanton induced two-point functions and anomalous dimensions

Before analysing specific cases of scalar operators in section 5 we now describe the general strategy that will be followed in such calculations. As usual in instanton calculus in the evaluation of the moduli space integrals it is convenient to perform the fermionic integrals first. In the case of two-point functions these are particularly simple and their analysis will allow us to show the absence of instanton corrections to a large class of operators.

4.1 Two-point functions in the one-instanton sector

In the special case of two-point functions the semiclassical approximation takes the form ²

$$\langle \bar{\mathcal{O}}(x_1) \mathcal{O}(x_2) \rangle = \frac{\pi^{-4N} g_{\text{YM}}^{4N} e^{2\pi i \tau}}{(N-1)!(N-2)!} \int d\rho d^4 x_0 \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A d^{N-2} \nu^A d^{N-2} \bar{\nu}^A \rho^{4N-13} \\ e^{\frac{\pi^2}{16g_{\text{YM}}^2 \rho^2} \varepsilon_{ABCD} (\bar{\nu}^{[A} \nu^{B]})(\bar{\nu}^{[C} \nu^{D]})} \hat{\mathcal{O}}(x_1; x_0, \rho, \eta, \bar{\xi}, \nu, \bar{\nu}) \hat{\mathcal{O}}(x_2; x_0, \rho, \eta, \bar{\xi}, \nu, \bar{\nu}), \quad (38)$$

where we have not yet rewritten the fermionic variables ν_u^A and $\bar{\nu}^{Au}$ in terms of bosonic auxiliary variables. As already observed in the general discussion of section 2 we have to distinguish between the superconformal fermionic modes, η_α^A and $\bar{\xi}^{\dot{\alpha}A}$, and the remaining ones, ν_u^A and $\bar{\nu}^{Au}$: the former do not appear in the measure and must be soaked up by the operator insertions in order for (38) to yield a non vanishing result. The ‘non-exact’ modes of type ν_u^A and $\bar{\nu}^{Au}$ appear in the measure so that an explicit dependence on these variables in the integrand is not required, although in general the classical profiles of composite operators do depend on $\bar{\nu}^A \nu^B$ colour singlet bilinears. After translating the dependence on ν_u^A and $\bar{\nu}^{Au}$ into a dependence on the angular variables Ω^{AB} using the generating function as discussed in section 2 the two-point function (38) becomes

$$\langle \bar{\mathcal{O}}(x_1) \mathcal{O}(x_2) \rangle = \frac{c(g_{\text{YM}}, N) g_{\text{YM}}^8 e^{2\pi i \tau}}{(N-1)!(N-2)!} \int d\rho d^4 x_0 d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \rho^{4N-7} \\ \hat{\mathcal{O}}(x_1; x_0, \rho, \eta, \bar{\xi}, \Omega) \hat{\mathcal{O}}(x_2; x_0, \rho, \eta, \bar{\xi}, \Omega), \quad (39)$$

where the coefficient $c(g_{\text{YM}}, N)$ contains additional dependence on g_{YM} and N arising from the integration over the radial variable r introduced in (13). As mentioned previously $c(g_{\text{YM}}, N)$ contains a power of g_{YM} for each $\bar{\nu}\nu$ bilinear in the integrand and a factor of \sqrt{N} plus $1/N$ corrections for each $(\bar{\nu}\nu)_6$.

As will be shown explicitly in the examples presented in section 5 the superconformal modes always appear in the expressions for gauge-invariant composite operators in the combination

$$\zeta_\alpha^A(x) = \frac{1}{\sqrt{\rho}} \left[\rho \eta_\alpha^A - (x - x_0)_\mu \sigma_{\alpha\dot{\alpha}}^\mu \bar{\xi}^{\dot{\alpha}A} \right]. \quad (40)$$

Since the two-component spinors $\zeta^A(x)$ satisfy $(\zeta^A(x))^3 = 0, \forall A, x$, it is clear that in a two-point function the only way of saturating the 16 integrations over η_α^A and $\bar{\xi}^{\dot{\alpha}A}$ in (39) is that each of the two operators provides the combination

$$[\zeta^1(x)]^2 [\zeta^2(x)]^2 [\zeta^3(x)]^2 [\zeta^4(x)]^2, \quad (41)$$

i.e. each of the two operators must soak up two powers of the superconformal combination (40) for of each of the four flavours. Unless this can be achieved the two-point function vanishes and therefore the anomalous dimension of the operator does not get instanton correction. This is a rather strong condition and it will allow us, using the results of section 3 to analyse the dependence on the superconformal modes, to show the absence of instanton corrections in many cases.

²In this general discussion we omit overall numerical constants in the integration measure.

If the operators can indeed soak up the right combination of superconformal modes the η and ξ integrals are non-vanishing and can be easily evaluated. Using a simple Fierz rearrangement one finds

$$\int d^2\eta^A d^2\bar{\xi}^A [\zeta^A(x_1)]^2 [\zeta^A(x_2)]^2 = -(x_1 - x_2)^2, \quad \forall A = 1, \dots, 4. \quad (42)$$

Once the integrations over the fermion superconformal modes have been performed we are left with an integration over the five-sphere parametrised by the angular variables Ω^{AB} and the bosonic part of the moduli space integration over x_0 and ρ . The five-sphere integration factorises and gives rise to further selection rules. It gives a non-vanishing result only if the SU(4) indices carried by the Ω 's in the two operators can be combined to form a SU(4) singlet. This analysis of the fermionic moduli space integrations can be repeated in the case of contributions with contractions between pairs of fields in the operator insertions. The same selection rules apply in these cases.

Before discussing specific examples we shall now briefly recall how the information on anomalous dimensions is encoded in the two-point functions and how it can be extracted from the final bosonic integration over the moduli space.

4.2 Anomalous dimensions

Gauge invariant composite operators in the $\mathcal{N}=4$ theory are classified in terms of their transformation under the PSU(2,2|4) supergroup of global symmetries. Each operator is characterised by a set of quantum numbers identifying the irreducible representation of the maximal bosonic subgroup SO(2,4) \times SU(4) of PSU(2,2|4) it transforms in. The quantum numbers defining such irreducible representations are $(\Delta, J_1, J_2; [a, b, c])$, where the spins J_1 and J_2 characterise the Lorentz group transformation and together with the scaling dimension Δ determine the transformation under the conformal group SO(2,4) and the remaining three numbers, $[a, b, c]$ are Dynkin labels of SU(4).

Let us consider a sector in the $\mathcal{N}=4$ theory formed by operators belonging to the same SU(4) and SO(1,3) representations (in the following we shall restrict our attention to Lorentz scalars) and with the same bare dimension, Δ_0 . In the quantum theory in general the operators acquire an anomalous dimension which corrects the bare value, $\Delta = \Delta_0 + \gamma$, where the anomalous term is a function of the parameters g_{YM} and N . Let us consider in such sector a complete set of n primary operators forming an orthonormal basis with respect to the scalar product defined by the two-point functions. We shall suppress Lorentz and SU(4) labels and denote the operators by $\mathcal{O}_{\Delta}^r(x)$. We assume that the operators are the ones well defined, *i.e.* transforming properly under all the global symmetries, in the full quantum theory. As is well known the spatial dependence in two-point functions of primary operators is completely fixed by conformal invariance. For the sector we are considering we have

$$\langle \bar{\mathcal{O}}_{\Delta_r}^r(x_1) \mathcal{O}_{\Delta_s}^s(x_2) \rangle = \frac{\delta^{rs}}{(x_1 - x_2)^{2\Delta_r}}. \quad (43)$$

In the full quantum theory $\Delta_r = \Delta_r(g_{\text{YM}}, N) = \Delta_0 + \gamma_r(g_{\text{YM}}, N)$ and the two-point function is non-zero only if $\bar{\mathcal{O}}_{\Delta_r}^r(x_1)$ and $\mathcal{O}_{\Delta_s}^s(x_2)$ have the same dimension, Δ_r . We

assume that in case there are operators with the same anomalous dimension they are made orthogonal, so that the set $\{\mathcal{O}^r\}$ forms an orthonormal basis in the sector under investigation in this case as well.

We want to compare (43) with the result of a small coupling calculation in order to extract from the latter the information about the anomalous dimensions. At small g_{YM} we assume the anomalous dimension $\gamma(g, N)$ to be small and hence expand (43) as

$$\langle \bar{\mathcal{O}}_{\Delta_r}^r(x_1) \mathcal{O}_{\Delta_s}^s(x_2) \rangle = \frac{\delta^{rs}}{(x_1 - x_2)^{2\Delta_0}} \frac{1}{\mu^{2\gamma_r(g_{\text{YM}}, N)}} \left[1 - \gamma_r(g_{\text{YM}}, N) \log \left(\frac{x_1 - x_2}{\mu} \right)^2 + \dots \right], \quad (44)$$

where μ is an arbitrary length scale related to the renormalisation scale in the small g_{YM} calculation. Clearly physical quantities, as the anomalous dimension γ , do not depend on μ . Equation (44) is a small- γ expansion, further expanding $\gamma(g_{\text{YM}}, N)$ at small g_{YM} we get

$$\begin{aligned} \langle \bar{\mathcal{O}}_{\Delta_r}^r(x_1) \mathcal{O}_{\Delta_s}^s(x_2) \rangle &= \frac{c(g_{\text{YM}}, N) \delta^{rs}}{(x_1 - x_2)^{2\Delta_0}} \left[1 - g_{\text{YM}}^2 \gamma_r^{(1)}(N) \log \left(\frac{x_1 - x_2}{\mu} \right)^2 \right. \\ &\quad \left. + \dots - e^{2\pi i \tau} \gamma_r^{(\text{inst})} \log \left(\frac{x_1 - x_2}{\mu} \right)^2 + \dots \right], \end{aligned} \quad (45)$$

where only the leading perturbative and instanton-induced terms have been kept. At higher orders in the double expansion (small γ and small g_{YM}) the situation is more complicated and terms with different powers of logarithms appear at the same order in g_{YM} . The terms in (45) are sufficient for the purposes of the analysis to be carried on in the following sections.

Equation (45) is the small coupling expansion of the exact two-point function of primary operators in an orthonormal basis. When performing explicit calculations, in perturbation theory or semiclassically in an instanton background, one works with a set of independent operators of bare dimension Δ_0 , $\tilde{\mathcal{O}}_{\Delta_0}^r$, which are not in general orthonormal with respect to the scalar product defined by the two-point function. In extracting the physical information, *i.e.* the anomalous dimensions, one has thus to deal with the complications associated with operator mixing.

The result for a two-point function of operators $\tilde{\mathcal{O}}_{\Delta_0}^r$ including the first order contributions in perturbation theory and the leading semiclassical instanton term is of the form

$$\begin{aligned} \langle \tilde{\mathcal{O}}_{\Delta_0}^r(x_1) \tilde{\mathcal{O}}_{\Delta_0}^s(x_2) \rangle &= \frac{1}{(x_1 - x_2)^{2\Delta_0}} \left[T^{rs} - g_{\text{YM}}^2 L^{rs} \log \left(\frac{x_1 - x_2}{\mu} \right)^2 \right. \\ &\quad \left. + \dots - e^{2\pi i \tau} K^{rs} \log \left(\frac{x_1 - x_2}{\mu} \right)^2 + \dots \right], \end{aligned} \quad (46)$$

where we have denoted by T , L and K the matrices arising at tree-level, one loop and in the one-instanton sector. To make contact with the expansion (45) of the exact function

we rewrite (46) as

$$\langle \tilde{\mathcal{O}}^r_{\Delta_0}(x_1) \tilde{\mathcal{O}}^s_{\Delta_0}(x_2) \rangle = \frac{1}{(x_1 - x_2)^{2\Delta_0}} T^{rs'} \left[\mathbb{1}^{s's} - e^{2\pi i \tau} \Gamma^{s's} \log \left(\frac{x_1 - x_2}{\mu} \right)^2 \right], \quad (47)$$

where only the instanton part has been kept and we have defined $\Gamma^{rs} = (T^{-1}K)^{rs}$. In order to bring (47) into the form (45) we need to perform a change of basis which takes from $\tilde{\mathcal{O}}^r$ to \mathcal{O}^r . This is achieved by acting with a matrix M

$$\langle \tilde{\mathcal{O}}^r(x_1) \tilde{\mathcal{O}}^s(x_2) \rangle = M^{rr'} (M^*)^{s's'} \langle \tilde{\mathcal{O}}^{r'}_{\Delta_0}(x_1) \tilde{\mathcal{O}}^{s'}_{\Delta_0}(x_2) \rangle. \quad (48)$$

Substituting (47) and (45) into this relation we find the matrix identity

$$\text{diag}(\{\gamma_r^{(\text{inst})}\}) = M \Gamma M^\dagger, \quad (49)$$

which implies that the instanton contributions to the anomalous dimensions in the sector under investigation can be identified with the eigenvalues of the matrix $\Gamma^{rs} = (T^{-1}K)^{rs}$. As discussed in [26, 28, 29] the above steps define the dilation operator of the theory, \hat{D} . In this language Γ is therefore identified with the leading instanton correction to the dilation operator.

5 Non-perturbative contributions to anomalous dimensions of scalar operators

Having described the general strategy for the computation of anomalous dimensions from two-point correlation functions we shall now consider explicit examples of Lorentz scalar operators of bare dimension $\Delta_0 = 2, 3, 4, 5$.

We shall work with the normalisation of the fields which is standard in instanton calculus, in which the action is written with an overall factor of $1/g_{\text{YM}}^2$. The usual normalisation of perturbative calculations is recovered rescaling all the fields, $\Phi \rightarrow g_{\text{YM}} \Phi$, so that the kinetic terms become g_{YM} -independent, the cubic couplings have a factor of g_{YM} and the quartic couplings are proportional to g_{YM}^2 . With our choice of normalisation the propagators are proportional to g_{YM}^2 . We shall define composite operators in such a way that their correlation functions are independent of g_{YM} at tree-level. Therefore for an operator made of ℓ elementary fields, which we denote generically by Φ_a , $a = 1, \dots, \ell$, we adopt the normalisation

$$\mathcal{O}^{(\ell)} = \frac{N}{(g_{\text{YM}}^2 N)^{\ell/2}} \text{Tr}(\Phi_1 \dots \Phi_\ell), \quad (50)$$

which is easily verified to agree with the standard convention used in the AdS/CFT correspondence in which two-point functions behave like N^2 at large N and are independent of the coupling at tree-level. The same rescaling that leads to the ordinary perturbative normalisation makes the composite operator (50) independent of g_{YM} .

The operators we focus on are Lorentz scalars and for fixed bare dimension, Δ_0 , they are characterised by their SU(4) Dynkin labels. We use the notation $\mathcal{O}_{\Delta_0, \mathbf{r}}$ to denote a scalar operator of bare dimension Δ_0 transforming in the \mathbf{r} -dimensional representation of SU(4). In each SU(4) sector we shall construct a complete set of independent operators and then study the instanton contributions to their two-point functions.

5.1 Dimension 2 scalar operators

At bare dimension 2 the situation is particularly simple. All operators are single trace and Lorentz scalars can only be obtained as bilinears in the elementary scalar fields,

$$\mathcal{O}_{2,\mathbf{6}\otimes\mathbf{6}}^{ij} \sim \text{Tr} \left(\varphi^i \varphi^j \right). \quad (51)$$

The scalars transform in the $\mathbf{6}$ of $\text{SU}(4)$ with Dynkin labels $[0, 1, 0]$ and thus the possible sectors for composite bilinears are those in the decomposition

$$[0, 1, 0] \otimes [0, 1, 0] = [0, 0, 0]_s \oplus [1, 0, 1]_a \oplus [0, 2, 0]_s \Leftrightarrow \mathbf{6} \otimes \mathbf{6} = \mathbf{1}_s \oplus \mathbf{15}_a \oplus \mathbf{20}'_s, \quad (52)$$

where the subscripts s and a indicate the symmetric and antisymmetric parts. There is actually no operator in the $\mathbf{15}$ because of cyclicity of the trace and the only two sectors of scalars with $\Delta_0 = 2$ are the singlet and $\mathbf{20}'$ with one operator in each of the two $\text{SU}(4)$ representations

$$[0, 0, 0] : \quad \mathcal{O}_{2,\mathbf{1}} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\varphi^i \varphi^i \right) \quad (53)$$

$$[0, 2, 0] : \quad \mathcal{O}_{2,\mathbf{20}'}^{\{ij\}} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\varphi^i \varphi^j - \frac{\delta^{ij}}{6} \varphi^k \varphi^k \right), \quad (54)$$

Here and in the following we indicate complete symmetrisation plus removal of traces by curly brackets, $\{i_1 i_2 \dots i_n\}$. We shall denote complete antisymmetrisation by square brackets, $[i_1 i_2 \dots i_n]$ and symmetrisation without removal of traces by parenthesis $(i_1 i_2 \dots i_n)$.

The two operators in (53) and (54) are of course well known. The singlet is the lowest component of the Konishi multiplet, $\mathcal{O}_{2,\mathbf{1}} \equiv \mathcal{K}_1$, and $\mathcal{O}_{2,\mathbf{20}'}^{ij} \equiv \mathcal{Q}^{ij}$ is the lowest component of the supercurrent multiplet containing the energy-momentum tensor and the supersymmetry and R-symmetry currents. The latter is a 1/2 BPS operator and thus its scaling dimension does not receive quantum corrections. Operators in the Konishi multiplet do have anomalous dimension in perturbation theory, but not at the instanton level. The one-loop contribution was first computed in [50] and re-derived in [4] using OPE techniques in the context of the AdS/CFT correspondence. The perturbative result has been extended to two-loops in [51, 52] and a three loop result has been obtained in [29] under certain assumptions studying the action of the dilation operator.

The fact that the anomalous dimension of the Konishi multiplet does not receive instanton corrections has been shown through the OPE analysis of a four-point function of 1/2 BPS operators \mathcal{Q}^{ij} in [4]. The derivation of this result directly from the study of the two-point function is rather subtle. Rewriting the scalar fields as $\varphi^{[AB]}$, \mathcal{K}_1 takes the form

$$\mathcal{K}_1 = \frac{1}{g_{\text{YM}}^2} \varepsilon_{ABCD} \text{Tr} \left(\varphi^{AB} \varphi^{CD} \right). \quad (55)$$

As discussed in section 4 in computing the instanton induced two-point function $\langle \mathcal{K}_1(x_1) \mathcal{K}_1(x_2) \rangle$ we integrate over the moduli space the classical expression of the operators in the background of an instanton and the superconformal fermionic integrations must be saturated. Each of the two operators must soak up eight fermion modes in the

combination (41). In order to get the correct number of fermion modes for each of the two insertions we need to consider

$$\varepsilon_{ABCD} \text{Tr} \left(\varphi^{(2)AB} \varphi^{(6)CD} + \varphi^{(6)AB} \varphi^{(2)CD} \right). \quad (56)$$

This combination contains the minimal required number of fermionic modes. The other terms in the scalar solution, $\varphi^{(10)AB}$ and $\varphi^{(14)AB}$, would give rise to contributions involving $\bar{\nu}\nu$ bilinears as well and so would be of higher order in g_{YM} and hence not relevant in semiclassical approximation. Using (32) and (37) the classical profile of \mathcal{K}_1 is seen to be proportional to

$$\varepsilon_{ABCD} \varepsilon_{A'B'C'D'} m_f^{[A} m_f^{B]} m_f^{[C} m_f^{D]} m_f^{A'} m_f^{B'} m_f^{C'} m_f^{D'}. \quad (57)$$

This expression indeed contains the combination (41). This means that the vanishing of the instanton contribution to the two-point function of the \mathcal{K}_1 operator and thus of its instanton induced anomalous dimension does not simply follow from the impossibility of saturating the fermionic integrals. The direct calculation of the (vanishing) anomalous dimension of \mathcal{K}_1 from the two-point function requires a cancellation among various terms for which we need the exact form of $\varphi^{(6)AB}$, which we have not determined. We shall find in section 5.3 that the absence of instanton correction for this operator can be shown analysing a superconformal descendant of \mathcal{K}_1 transforming in the representation **84**.

For the 1/2 BPS operator $\mathcal{Q}^{ij} = \mathcal{O}_{2,20'}^{ij}$, on the other hand, we can easily verify the absence of instanton corrections to the two-point function. For this purpose it is convenient to choose a specific component to evaluate the two-point function, *e.g.*

$$\langle \mathcal{Q}^{12}(x_1) \mathcal{Q}^{12}(x_2) \rangle. \quad (58)$$

Recalling that $\varphi^1 = \sqrt{2}(\varphi^{14} + \varphi^{23})$ and $\varphi^2 = \sqrt{2}(\varphi^{24} + \varphi^{13})$ (see appendix A) we find that in the instanton background the operator \mathcal{Q}^{12} is proportional to

$$\begin{aligned} \varepsilon_{A'B'C'D'} m_f^{A'} m_f^{B'} m_f^{C'} m_f^{D'} \left(m_f^{[1} m_f^{4]} m_f^{[2} m_f^{4]} + m_f^{[2} m_f^{3]} m_f^{[2} m_f^{4]} \right. \\ \left. + m_f^{[1} m_f^{4]} m_f^{[1} m_f^{3]} + m_f^{[2} m_f^{3]} m_f^{[1} m_f^{3]} \right), \end{aligned} \quad (59)$$

and taking all the modes to be superconformal modes none of the four terms in this expression contains the combination (41).

The above detailed discussion was clearly not needed for \mathcal{Q}^{ij} which is known to be protected, as already remarked, but it is included in order to illustrate the strategy utilised in more complicated examples in the following sections.

The operator $\mathcal{O}_{2,20'}^{ij}$ is the simplest example in a class of operators characterised by the fact that their bare dimension is $\Delta_0 = \ell$ and they transform in the representation $[0, \ell, 0]$ of $\text{SU}(4)$. Operators of this type which are superconformal primaries are 1/2 BPS [53, 54]. As shown in [29, 40] operators of this type form a ‘sector’ in the $\mathcal{N}=4$ SYM theory. They cannot mix with operators involving fermions, field strengths or covariant derivatives at any order in g_{YM} . For operators of this type it is always possible to choose a component which in $\mathcal{N}=1$ notation can be written in terms of a single complex scalar. This makes proving the absence of instanton corrections very simple. For \mathcal{Q}^{ij} we can consider the component $\text{Tr}(\phi^1 \phi^1) \sim \text{Tr}(\varphi^{14} \varphi^{14})$ (see appendix A for the definition of the complex combinations ϕ^I), so that it is immediately verified that it cannot saturate the eight fermion modes as in (41).

5.2 Dimension 3 scalar operators

Composite operators of bare dimension $\Delta_0 = 3$ are also necessarily single trace. They can be obtained either as products of three scalars or as fermion bilinears, which must involve spinors of the same chirality in order for the composite to be a Lorentz scalar. So we need to consider

$$\mathcal{O}_{3,6\otimes 6\otimes 6}^{ijk} \sim \text{Tr}(\varphi^i \varphi^j \varphi^k), \quad \mathcal{O}_{3,4\otimes 4}^{AB} \sim \text{Tr}(\lambda^A \lambda^B), \quad \mathcal{O}_{3,\bar{4}\otimes \bar{4};AB} \sim \text{Tr}(\bar{\lambda}_A \bar{\lambda}_B). \quad (60)$$

Operators cubic in the elementary scalars belong to the sectors in the decomposition

$$\begin{aligned} [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] &= 3[0, 1, 0] \oplus [2, 0, 0] \oplus [0, 0, 2] \oplus [0, 3, 0] \oplus 2[1, 1, 1] \\ \Leftrightarrow \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} &= 3 \cdot \mathbf{6} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{50} \oplus 2 \cdot \mathbf{64}, \end{aligned} \quad (61)$$

whereas the fermionic bilinears contribute to the sectors

$$[1, 0, 0] \otimes [1, 0, 0] = [0, 1, 0]_a \oplus [2, 0, 0]_s \Leftrightarrow \mathbf{4} \otimes \mathbf{4} = \mathbf{6}_a \oplus \mathbf{10}_s \quad (62)$$

$$[0, 0, 1] \otimes [0, 0, 1] = [0, 1, 0]_a \oplus [0, 0, 2]_s \Leftrightarrow \bar{\mathbf{4}} \otimes \bar{\mathbf{4}} = \mathbf{6}_a \oplus \overline{\mathbf{10}}_s. \quad (63)$$

5.2.1 $\Delta_0 = 3, [0, 1, 0]$

In this sector there is only one operator coming from (61). The decomposition $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$ contains the $\mathbf{6}$ with multiplicity 3, but taking into account the cyclicity of the trace there is only one independent operator,

$$\mathcal{O}_{3,6}^i = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr}(\varphi^i \varphi^j \varphi^j). \quad (64)$$

The $\mathbf{6}$ is also contained in $\mathbf{4} \otimes \mathbf{4}$ and $\bar{\mathbf{4}} \otimes \bar{\mathbf{4}}$, but the corresponding fermionic bilinears vanish for the cyclicity of the trace,

$$\Sigma_{AB}^i \text{Tr}(\lambda^{\alpha A} \lambda_{\alpha}^B) = \bar{\Sigma}_i^{AB} \text{Tr}(\bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{\alpha}}^B) = 0, \quad (65)$$

since Σ_{AB}^i and $\bar{\Sigma}_i^{AB}$ are antisymmetric in A, B .

Let us then consider the two-point function of operators $\mathcal{O}_{3,6}^i$. For definiteness we take the component $i = 1$, the correlator $\langle \mathcal{O}_{3,6}^i(x_1) \mathcal{O}_{3,6}^j(x_2) \rangle$ is clearly proportional to δ^{ij} . The first step is to check whether each of the two operators can saturate the superconformal fermion integrations, *i.e.* whether their instanton profile contains the combination (41). We need to consider

$$\begin{aligned} &\text{Tr}(\varphi^{(2)1} \varphi^{(2)j} \varphi^{(6)j} + \varphi^{(2)1} \varphi^{(6)j} \varphi^{(2)j} + \varphi^{(6)1} \varphi^{(2)j} \varphi^{(2)j}) \\ &\sim \varepsilon_{ABCD} \varepsilon_{A'B'C'D'} \left[\left(m_{\text{f}}^{[1} m_{\text{f}}^{4]} + m_{\text{f}}^{[2} m_{\text{f}}^{3]} \right) \left(m_{\text{f}}^{[A} m_{\text{f}}^{B]} \right) \left(m_{\text{f}}^{[C} m_{\text{f}}^{D]} \right) \left(m_{\text{f}}^{A'} m_{\text{f}}^{B'} m_{\text{f}}^{C'} m_{\text{f}}^{D'} \right) \right], \end{aligned} \quad (66)$$

where we have used (32) and (37) and written the contraction $\varphi^j \varphi^j$ as $\varepsilon_{ABCD} \varphi^{AB} \varphi^{CD}$. Inspecting (66) it is easily verified that indeed the superconformal modes can be saturated in the two-point function. A potentially non-vanishing contribution is obtained for instance from the first term in (66) replacing φ^1 by $\bar{\nu} \nu$ bilinears and using the remaining

scalars to soak up the 16 ζ 's in each of the two operators. The other terms give identical contributions up the overall coefficient. The resulting contribution to the two-point function is schematically of the form

$$\langle \mathcal{O}_{3,6}^1(x_1) \mathcal{O}_{3,6}^1(x_2) \rangle \sim \int d\mu_{\text{phys}} e^{-S_{\text{inst}}} \quad (67)$$

$$f(x_1; x_0, \rho) \left(\bar{\nu}^{[1} \nu^{4]} \right) \prod_{A=1}^4 \left[\zeta^A(x_1) \right]^2 f(x_2; x_0, \rho) \left(\bar{\nu}^{[2} \nu^{3]} \right) \prod_{A'=1}^4 \left[\zeta^{A'}(x_2) \right]^2 ,$$

where the function $f(x; x_0, \rho)$ depends on the details of the solutions $\varphi^{(2)}$ and $\varphi^{(6)}$ for the scalars. Writing the measure explicitly as in section 2 to we get

$$\begin{aligned} \langle \mathcal{O}_{3,6}^1(x_1) \mathcal{O}_{3,6}^1(x_2) \rangle &\sim \frac{g_{\text{YM}}^2 e^{2\pi i \tau}}{N(N-1)!(N-2)!} \int d\rho d^4 x_0 d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \rho^{4N-7} \\ &\quad \int_0^\infty dr r^{4N-3} e^{-2\rho^2 r^2} f(x_1; x_0, \rho) \prod_{A=1}^4 \left[\zeta^A(x_1) \right]^2 f(x_2; x_0, \rho) \prod_{A'=1}^4 \left[\zeta^{A'}(x_2) \right]^2 \\ &\quad \frac{\delta^4}{\delta \bar{\vartheta}_{[1}^u \delta \vartheta_{u4]} \delta \bar{\vartheta}_{[2}^v \delta \vartheta_{v3]} \mathcal{Z}(\vartheta, \bar{\vartheta}; \Omega, r) \Big|_{\vartheta=\bar{\vartheta}=0}} \\ &\sim \frac{g_{\text{YM}}^2 \Gamma(2N-2) 2^{-2N} e^{2\pi i \tau}}{(N-1)!(N-2)!} (x_1 - x_2)^8 \int \frac{d\rho d^4 x_0}{\rho^3} f(x_1; x_0, \rho) f(x_2; x_0, \rho) \\ &\quad \int d^5 \Omega \left[(N-2)^2 \Omega^{14} \Omega^{23} - (N-2) \Omega^{13} \Omega^{24} \right] . \end{aligned} \quad (68)$$

The five-sphere integral is evaluated using

$$\int d^5 \Omega \Omega^{AB} \Omega^{CD} = \frac{1}{4} \varepsilon^{ABCD} \quad (69)$$

and we finally find

$$\langle \mathcal{O}_{3,6}^1(x_1) \mathcal{O}_{3,6}^1(x_2) \rangle \sim g_{\text{YM}}^2 c(N) e^{2\pi i \tau} (x_1 - x_2)^8 \int \frac{d\rho}{\rho^3} d^4 x_0 f(x_1; x_0, \rho) f(x_2; x_0, \rho) , \quad (70)$$

where for large N $c(N) = N\sqrt{N}[1 + O(1/N)]$. The complete result for the two-point function is a sum of terms of this form. A logarithmic divergence in the remaining bosonic integrals would signal an instanton contribution to the anomalous dimension of $\mathcal{O}_{3,6}^i$. Since we have not determined the complete expression for $\varphi^{(6)}$ we cannot compute the exact coefficient in (70) and verify the presence of the divergence.

Notice however that (70) contains, apart from the standard instanton weight, a factor of $\lambda = g_{\text{YM}}^2 N$. Since in this sector there is only one operator there is no problem of mixing to take into account and the instanton induced anomalous dimension can be read directly from the two-point function (70) (after dividing by the tree-level coefficient). The result is that if $\mathcal{O}_{3,6}^i$ does acquire an anomalous dimension, it is of order

$$\gamma_{3,6}^{\text{inst}} \sim \lambda e^{2\pi i \tau} . \quad (71)$$

This is to be contrasted with the case of other operators that we shall examine in the following in which the leading instanton contribution to two-point functions behaves as $e^{2\pi i \tau}$ with no further factors of g_{YM} .

We shall come back to the case of the operator $\mathcal{O}_{3,6}^i$ in section 5.2.5.

5.2.2 $\Delta_0 = 3$, $[2, 0, 0]$ and $[0, 0, 2]$

There are two scalar operators with $\Delta_0 = 3$ in the $[2, 0, 0]$, one is cubic in the scalars and the other is bilinear in the $[1, 0, 0]$ fermions,

$$\mathcal{O}_{3,10}^{(1)(AB)} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} t_{[ijk]}^{(AB)} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \right) \quad (72)$$

$$\mathcal{O}_{3,10}^{(2)(AB)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\lambda^{\alpha A} \lambda_{\alpha}^B \right), \quad (73)$$

where the projector $t_{[ijk]}^{(AB)}$ of the product $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$ onto the $\mathbf{10}$ is defined as

$$t_{\mathbf{10}[ijk]}^{(AB)} = \bar{\Sigma}_{[i}^{AC} \Sigma_{jCD} \bar{\Sigma}_{k]}^{DB}, \quad (74)$$

with the brackets indicating complete antisymmetrisation in i, j and k . A similar projector onto the $\overline{\mathbf{10}}$ can be constructed as

$$\bar{t}_{\overline{\mathbf{10}}(AB)}^{[ijk]} = \Sigma_{AC}^{[i} \bar{\Sigma}^{jCD} \Sigma_{DB}^{k]}, \quad (75)$$

so that in the $[0, 0, 2]$ we find

$$\bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(AB)}^{(1)} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \bar{t}_{(AB)}^{[ijk]} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \right) \quad (76)$$

$$\bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(AB)}^{(2)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\bar{\lambda}_{\dot{\alpha}A} \bar{\lambda}_{\dot{\alpha}B} \right). \quad (77)$$

We now consider the two-point functions in this sector. In order to saturate the fermionic integrals over the superconformal modes we need to consider the following terms

$$\mathcal{O}_{3,10}^{(1)(AB)} \sim t_{[ijk]}^{(AB)} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \varphi^{(6)k} + (\varphi^{(2)i} \varphi^{(6)j} \varphi^{(2)k}) + (\varphi^{(6)i} \varphi^{(2)j} \varphi^{(2)k}) \right) \quad (78)$$

$$\mathcal{O}_{3,10}^{(2)(AB)} \sim \text{Tr} \left(\lambda^{(5)\alpha A} \lambda_{\alpha}^{(5)B} + \lambda^{(1)\alpha A} \lambda_{\alpha}^{(9)B} + \lambda^{(9)\alpha A} \lambda_{\alpha}^{(1)B} \right) \quad (79)$$

and similarly for the conjugate operators we need

$$\bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(AB)}^{(1)} \sim \bar{t}_{(AB)}^{[ijk]} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \varphi^{(6)k} + (\varphi^{(2)i} \varphi^{(6)j} \varphi^{(2)k}) + (\varphi^{(6)i} \varphi^{(2)j} \varphi^{(2)k}) \right) \quad (80)$$

$$\bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(AB)}^{(2)} \sim \text{Tr} \left(\bar{\lambda}_{\dot{\alpha}A}^{(3)} \bar{\lambda}_{\dot{\alpha}B}^{(7)} + \bar{\lambda}_{\dot{\alpha}A}^{(7)} \bar{\lambda}_{\dot{\alpha}B}^{(3)} \right). \quad (81)$$

As usual it is convenient to pick a specific component, so *e.g.* we consider $\langle \mathcal{O}_{3,10}^{(r)(11)}(x_1) \bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(11)}^{(s)}(x_2) \rangle$, with $r, s = 1, 2$. The two operators $\mathcal{O}_{3,10}^{(r)(11)}$ are

$$\mathcal{O}_{3,10}^{(1)(11)} = \frac{9\sqrt{8}}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\varphi^{12} [\varphi^{13}, \varphi^{14}] \right), \quad \mathcal{O}_{3,10}^{(2)(11)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\lambda^{\alpha 1} \lambda_{\alpha}^1 \right) \quad (82)$$

and their conjugates are

$$\bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(11)}^{(1)} = \frac{9\sqrt{8}}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\varphi^{23} [\varphi^{24}, \varphi^{34}] \right), \quad \bar{\mathcal{O}}_{3,\overline{\mathbf{10}}(11)}^{(2)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\bar{\lambda}_{\dot{\alpha}1} \bar{\lambda}_{\dot{\alpha}1} \right). \quad (83)$$

Recalling the analysis of section 3 we find that the two operators $\mathcal{O}_{3,10}^{(r)(11)}$ can soak up the fermionic modes in the combination required to get a non-zero contribution to the two-point function. Expanding (82) as in (78)-(79) we obtain terms which contain the combination (41) and are proportional to

$$(\bar{\nu}^1 \nu^1) [\zeta^1(x)]^2 [\zeta^2(x)]^2 [\zeta^3(x)]^2 [\zeta^4(x)]^2. \quad (84)$$

None of the conjugate operators however can provide (41), they can at most contain one mode ζ^1 . Hence the two-point functions are all zero in the instanton background and the scaling dimensions do not receive instanton corrections in this sector.

Notice that even before analysing the conjugate operators we could conclude that the two-point functions are not corrected by instantons. From (84) it is clear that the five-sphere integration would vanish as a consequence of (69).

These results were to be expected. Two orthogonal operators of dimension 3 in the **10** can be defined as

$$\mathcal{E}^{(AB)} = -\frac{1}{g_{\text{YM}}^2} \text{Tr}(\lambda^{\alpha A} \lambda_{\alpha}^B) + \frac{\sqrt{2}}{g_{\text{YM}}^2} t_{[ijk]}^{(AB)} \text{Tr}(\varphi^i \varphi^j \varphi^k) \quad (85)$$

$$\mathcal{K}^{(AB)} = \frac{3\sqrt{2}}{g_{\text{YM}}^2} t_{[ijk]}^{(AB)} \text{Tr}(\varphi^i \varphi^j \varphi^k) + \frac{6N}{32\pi^2} \text{Tr}(\lambda^{\alpha A} \lambda_{\alpha}^B). \quad (86)$$

$\mathcal{E}^{(AB)}$ is a component of the 1/2 BPS multiplet of the $\mathcal{N}=4$ supercurrents, it is found at level δ^2 starting from the lowest component $\mathcal{O}_{2,20'}^{\{ij\}}$, and $\mathcal{K}^{(AB)}$ is a component at the same level of the Konishi multiplet. Therefore the former is protected and the latter does not receive instanton corrections. Notice that the relative coefficients in (85) can be fixed for instance by the requiring that the three-point function $\langle \mathcal{E}^{(AB)} \mathcal{E}^{(CD)} \mathcal{Q}^{ij} \rangle$ does not receive perturbative corrections [55], whereas the coefficients in (86) are determined by the Konishi anomaly [38].

5.2.3 $\Delta_0 = 3, [0, 3, 0]$

In this sector there is only one operator which is obtained from the fully symmetrised product of three scalars

$$\begin{aligned} \mathcal{O}_{3,50}^{\{ijk\}} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr}(\varphi^{\{i} \varphi^j \varphi^{k\}}) \\ &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \left[\text{Tr}(\varphi^i \varphi^j \varphi^k + \varphi^i \varphi^k \varphi^j) - \frac{1}{4} \text{Tr}(\delta^{ij} \varphi^k \varphi^l \varphi^l + \delta^{ik} \varphi^j \varphi^l \varphi^l + \delta^{jk} \varphi^i \varphi^l \varphi^l) \right]. \end{aligned} \quad (87)$$

This sector is the next example of the type discussed at the end of section 5.1. The operator in (87) is a protected 1/2 BPS operator and using the notation of section 5.1 we define $\mathcal{Q}^{ijk} \equiv \mathcal{O}_{3,50}^{\{ijk\}}$. The 1/2 BPS operator in the **20'** is the lowest component of the supercurrent multiplet. It is dual in the AdS/CFT correspondence to a scalar in the type IIB supergravity multiplet which is a linear combination of the trace part of the metric and the $R \otimes R$ 4-form with indices in internal directions. The operator \mathcal{Q}^{ijk} is dual to

the first Kaluza–Klein excited mode of the same field ³. It is straightforward to verify the absence of instanton corrections to two point-functions of \mathcal{Q}^{ijk} by choosing a suitable component to analyse the zero-mode structure. The simplest choice is a component written in terms of a single complex scalar, $\text{Tr}(\phi^1\phi^1\phi^1)$. The terms in this operator with at least eight fermion modes in the instanton background are

$$\text{Tr}\left(\phi^{(6)1}\phi^{(2)1}\phi^{(2)1}\right) \sim \varepsilon_{ABCD} \left(m_f^{[1} m_f^{4]} m_f^{[1} m_f^{4]} m_f^{[1} m_f^{4]} m_f^A m_f^B m_f^C m_f^D\right), \quad (88)$$

from which one immediately verifies that only one mode of flavours 2 and 3 can be saturated, so that the two-point functions of this operator cannot get instanton correction.

5.2.4 $\Delta_0 = 3, [1, 1, 1]$

Operators with $\Delta_0 = 3$ in the **64**, corresponding to Dynkin labels $[1, 1, 1]$, also involve only elementary scalars. The **64** occurs twice in the product $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$, respectively in $\mathbf{6} \otimes \mathbf{15}$ and $\mathbf{6} \otimes \mathbf{20}'$. To obtain the operators in the **64** we should suitably project the two combinations

$$\text{Tr}\left(\varphi^i \varphi^{[j} \varphi^{k]}\right) \quad (89)$$

$$\text{Tr}\left(\varphi^i \varphi^{\{j} \varphi^{k\}}\right). \quad (90)$$

Taking into account the cyclicity of the trace one finds that (89) actually only contributes to the already examined **10** and $\overline{\mathbf{10}}$, since the operator is automatically fully antisymmetric in i, j and k . Similarly (90) is found to only contribute to the **50** and **6** representations: it is not possible to make (90) orthogonal to both (64) and (87). Hence the **64**, although allowed by the group theoretical analysis, is not realised in terms of gauge invariant operators.

5.2.5 $\Delta_0 = 3, [0, 1, 0]$ revisited

The non-renormalisation results for operators in the representations **10**, $\overline{\mathbf{10}}$ and **50** (and the absence of gauge-invariant operators in the **64**) simplify the computation of the anomalous dimension of $\mathcal{O}_{3,6}^i$ using an alternative approach, based on the OPE analysis of a four-point function. This method is more complicated, because it involves the computation of an instanton induced four-point function, but does not require to solve the field equation for $\varphi^{(6)}$.

A four-point function of protected operators \mathcal{Q} can be expanded in a double OPE as

$$\langle \mathcal{Q}^r(x_1) \mathcal{Q}^s(x_1) \mathcal{Q}^u(x_3) \mathcal{Q}^v(x_4) \rangle = \sum_k \frac{C_k^{rs}(x_{12}, \partial_1) C_k^{uv}(x_{34}, \partial_3)}{(x_{12})^{\Delta_r + \Delta_s - \Delta_k} (x_{34})^{\Delta_u + \Delta_v - \Delta_k}} \langle \mathcal{O}^k(x_1) \mathcal{O}^k(x_3) \rangle, \quad (91)$$

where $x_{ij} = x_i - x_j$, $\partial_i = \partial/\partial x_i$ and the sum is over a generally infinite set of primary operators. The dependence on derivatives in the Wilson coefficients C_k^{rs} implicitly represents the inclusion of descendants. The operators \mathcal{Q} are chosen to be protected, but

³In this sense the operator $\mathcal{O}_{3,6}^i$ of equation (64) can be thought of as the “first Kaluza–Klein excitation” of the Konishi operator, \mathcal{K}_1 , considered in section 5.1.

in general some of the \mathcal{O}_k 's may have anomalous dimension, so that $\Delta_k = \Delta_k^{(0)} + \gamma_k$. Similarly for the C_k^{rs} coefficients we have $C_k^{rs} = C_k^{(0)rs} + \omega_k^{rs}$. Expanding (91) for small γ_k and ω_k^{rs} we get

$$\langle \mathcal{Z}^r(x_1) \mathcal{Z}^s(x_1) \mathcal{Z}^u(x_3) \mathcal{Z}^v(x_4) \rangle = \sum_k \frac{\langle \mathcal{O}^k(x_1) \mathcal{O}^k(x_3) \rangle_{(0)}}{(x_{12})^{\Delta_r + \Delta_s - \Delta_k^{(0)}} (x_{34})^{\Delta_u + \Delta_v - \Delta_k^{(0)}}} \left[C_k^{(0)rs} C_k^{(0)uv} + C_k^{(0)rs} \omega_k^{uv} + \omega_k^{rs} C_k^{(0)uv} + \frac{\gamma_k}{2} C_k^{(0)rs} C_k^{(0)uv} \log \left(\frac{x_{12}^2 x_{34}^2}{x_{13}^4} \right) \right], \quad (92)$$

which shows how anomalous dimensions and corrections to the OPE coefficients can be extracted from the analysis of four-point functions.

In order to have the exchange of the a scalar in the **6** we can consider the instanton contribution to the correlation function

$$G(x_1, x_2, x_3, x_4) = \langle \mathcal{O}_{3,50}(x_1) \mathcal{O}_{2,20'}(x_2) \mathcal{O}_{3,50}(x_3) \mathcal{O}_{2,20'}(x_4) \rangle, \quad (93)$$

involving two $\Delta_0 = 2$ and two $\Delta_0 = 3$ 1/2 BPS operators. In computing (93) at leading order we only need the scalar solution $\varphi^{(2)}$ which is given in appendix B.

The operators exchanged in the s -channel, corresponding to $x_{12} \rightarrow 0$, $x_{34} \rightarrow 0$, are in the decomposition

$$\mathbf{20}' \otimes \mathbf{50} = \mathbf{6} \oplus \mathbf{50} \oplus \mathbf{64} \oplus \mathbf{196} \oplus \mathbf{300} \oplus \mathbf{384}. \quad (94)$$

According to the general formula (92) the presence of an instanton induced anomalous dimension for an operator of dimension 3 would give rise to a singularity of the type

$$\frac{1}{(x_{12})^2 (x_{34})^2} \log \left(\frac{(x_{12})^2 (x_{34})^2}{(x_{13})^4} \right). \quad (95)$$

Since we have verified that $\mathcal{O}_{3,6}^i$ is the only operator of bare dimension 3 which can have contribution at the instanton level, the OPE analysis of a single four-point function unambiguously determines its anomalous dimension. It can be read off from the coefficient of the singular term (95) with no problem of mixing to take into account and no necessity to project onto the representation we are interested in. This is the same type of argument used in [4] to show that the anomalous dimension of the Konishi operator, \mathcal{K}_1 , is not corrected non-perturbatively. In that case the argument took advantage of the fact that \mathcal{K}_1 is the only operator of bare dimension 2 which could possibly be corrected.

The computation of the correlator (93) in the one-instanton sector is presented in appendix C. As shown there in the limit $x_{12} \rightarrow 0$, $x_{34} \rightarrow 0$ this four-point function does not have a singularity of the type (95), indicating that the operator $\mathcal{O}_{3,6}^i$ does not have anomalous dimension at the instanton level. This means that the coefficient of the logarithmically divergent term in the two-point function (70) actually vanishes.

5.3 Dimension 4 scalar operators

The analysis operators of bare dimension $\Delta_0 = 4$ is more involved, at this level derivatives and field strengths also contribute to scalar operators and moreover we need to consider single- as well as double-trace operators.

The combinations of elementary fields we need to consider here are

$$\mathcal{O}_{4,1}^{(a)} \sim \text{Tr} (F_{\mu\nu} F^{\mu\nu}) , \quad \mathcal{O}_{4,1}^{(b)} \sim \text{Tr} (F_{\mu\nu} \tilde{F}^{\mu\nu}) \quad (96)$$

$$\mathcal{O}_{4,6\otimes 6}^i \sim \text{Tr} (\mathcal{D}_\mu \varphi^i \mathcal{D}^\mu \varphi^j) \quad (97)$$

$$\mathcal{O}_{4,6\otimes 4\otimes 4}^{iAB} \sim \text{Tr} (\varphi^i \lambda^{\alpha A} \lambda_\alpha^B) , \quad \mathcal{O}_{AB;4,6\otimes \bar{4}\otimes \bar{4}}^i \sim \text{Tr} (\varphi^i \bar{\lambda}_{\dot{\alpha} A} \lambda_{\dot{B}}^{\dot{\alpha}}) \quad (98)$$

$$\mathcal{O}_{4,6\otimes 6\otimes 6\otimes 6}^{(s)ijkl} \sim \text{Tr} (\varphi^i \varphi^j \varphi^k \varphi^l) , \quad \mathcal{O}_{4,6\otimes 6\otimes 6\otimes 6}^{(d)ijkl} \sim \text{Tr} (\varphi^i \varphi^j) \text{Tr} (\varphi^k \varphi^l) , \quad (99)$$

where the notation is the same used in previous sections and superscripts (s) and (d) denote single- and double-trace operators. This shows that operators involving field strengths can only contribute to the SU(4) singlet sector. Operators with covariant derivatives can be found in the sectors in the decomposition

$$[0, 1, 0] \otimes [0, 1, 0] = [0, 0, 0] \oplus [1, 0, 1] \oplus [0, 2, 0] \Leftrightarrow \mathbf{6} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20}' . \quad (100)$$

The operators of the two types in (98), made out of a scalar and a fermion bilinear, contribute respectively to

$$\begin{aligned} [0, 1, 0] \otimes [1, 0, 0] \otimes [1, 0, 0] &= [0, 0, 0] \oplus 2[1, 0, 1] \oplus [0, 2, 0] \oplus [2, 1, 0] \\ &\Leftrightarrow \mathbf{6} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus 2 \cdot \mathbf{15} \oplus \mathbf{20}' \oplus \mathbf{45} \end{aligned} \quad (101)$$

and

$$\begin{aligned} [0, 1, 0] \otimes [0, 0, 1] \otimes [0, 0, 1] &= [0, 0, 0] \oplus 2[1, 0, 1] \oplus [0, 2, 0] \oplus [0, 1, 2] \\ &\Leftrightarrow \mathbf{6} \otimes \bar{\mathbf{4}} \otimes \bar{\mathbf{4}} = \mathbf{1} \oplus 2 \cdot \mathbf{15} \oplus \mathbf{20}' \oplus \bar{\mathbf{45}} . \end{aligned} \quad (102)$$

The single- and double-trace operators in (99) can contribute to the representations in

$$\begin{aligned} [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] &= 3[0, 0, 0] \oplus 7[1, 0, 1] \oplus 6[0, 2, 0] \\ &\quad \oplus 3([2, 1, 0] \oplus [0, 1, 2]) \oplus 2[2, 0, 2] \oplus [0, 4, 0] \oplus [1, 2, 1] \\ &\Leftrightarrow \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = 3 \cdot \mathbf{1} \oplus 7 \cdot \mathbf{15} \oplus 6 \cdot \mathbf{20}' \oplus 3 \cdot (\mathbf{45} \oplus \bar{\mathbf{45}}) \oplus 2 \cdot \mathbf{84} \oplus \mathbf{105} \oplus \mathbf{175} . \end{aligned} \quad (103)$$

5.3.1 $\Delta_0 = 4$, $[0, 0, 0]$

In the singlet sector at $\Delta_0 = 4$ one can consider the following basis of operators.

There are the two operators (96) involving field strengths, which including the normalisation read

$$\mathcal{O}_{4,1}^{(1)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) \quad \text{and} \quad \mathcal{O}_{4,1}^{(2)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} (F_{\mu\nu} \tilde{F}^{\mu\nu}) . \quad (104)$$

With two derivatives there is the operator

$$\mathcal{O}_{4,1}^{(3)} = \frac{1}{g_{\text{YM}}^2} \text{Tr} (\mathcal{D}_\mu \varphi^i \mathcal{D}^\mu \varphi^i) . \quad (105)$$

The two singlets in (98) are respectively

$$\mathcal{O}_{4,1}^{(4)} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \Sigma_{AB}^i \text{Tr} (\varphi^i \lambda^{\alpha A} \lambda_\alpha^B) \quad \text{and} \quad \mathcal{O}_{4,1}^{(5)} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \bar{\Sigma}_i^{AB} \text{Tr} (\varphi^i \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{B}}^{\dot{\alpha}}) . \quad (106)$$

Finally using cyclicity of the trace one finds four (out of the possible six) operators made out of four scalars, two single-trace and two double-trace,

$$\mathcal{O}_{4,1}^{(6)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \varphi^i \varphi^j \right), \quad \mathcal{O}_{4,1}^{(7)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^i \varphi^j \varphi^j \right), \quad (107)$$

$$\mathcal{O}_{4,1}^{(8)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \right) \text{Tr} \left(\varphi^i \varphi^j \right), \quad \mathcal{O}_{4,1}^{(9)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^i \right) \text{Tr} \left(\varphi^j \varphi^j \right). \quad (108)$$

Among the $\Delta_0 = 4$ operators in the $\text{SU}(4)$ singlet two are known to be protected. They are the θ^4 and $\bar{\theta}^4$ components of the $\mathcal{N}=4$ supercurrent multiplet, \mathcal{C}^- and \mathcal{C}^+ . These are linear combinations of (104), (105), (106) and (107),

$$\mathcal{C}^- = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(F_{\mu\nu}^- F^{-\mu\nu} \right) + \dots, \quad \mathcal{C}^+ = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(F_{\mu\nu}^+ F^{+\mu\nu} \right) + \dots. \quad (109)$$

These two operators are 1/2 BPS and dual to complex combinations of the dilaton and $\text{R} \otimes \text{R}$ scalar in the AdS/CFT correspondence, τ and $\bar{\tau}$, where $\tau = \tau_1 + i\tau_2 = C^{(0)} + ie^{-\phi}$. The sum $\mathcal{C}^- + \mathcal{C}^+$ is proportional to the $\mathcal{N}=4$ on-shell lagrangian.

Another known operator in this sector is a component of the Konishi multiplet, \mathcal{K}'_1 , which is a different linear combination of the same operators, orthogonal to \mathcal{C}^- and \mathcal{C}^+ . Since it belongs to the Konishi multiplet this operators is not corrected at the instanton level.

The operators (107) and (108) have been studied in perturbation theory in [29, 58] and their anomalous dimensions were computed at one loop, where there is no mixing with the remaining operators in the sector. At higher loops mixing is expected to occur. At the non-perturbative level there is mixing among all the operators (104)-(108) at leading order in g_{YM} . As will be now shown all the operators $\mathcal{O}_{4,1}^{(r)}$, $r = 1, \dots, 9$, can soak up the correct combination of fermionic modes and hence all their two-point functions can get instanton corrections.

For the two operators (104) the terms contributing to two-point functions in the one-instanton sector are

$$\mathcal{O}_{4,1}^{(1)} \rightarrow \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(F_{\mu\nu}^{(4)} F^{(4)\mu\nu} \right) \quad (110)$$

$$\mathcal{O}_{4,1}^{(2)} \rightarrow \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(F_{\mu\nu}^{(4)} \tilde{F}^{(4)\mu\nu} \right), \quad (111)$$

which using (35) can be shown to be proportional to

$$\varepsilon_{ABCD} \varepsilon_{A'B'C'D'} \left(m_{\text{f}}^A m_{\text{f}}^B m_{\text{f}}^C m_{\text{f}}^D \right) \left(m_{\text{f}}^{A'} m_{\text{f}}^{B'} m_{\text{f}}^{C'} m_{\text{f}}^{D'} \right), \quad (112)$$

so that they contain the combination (41).

Similarly for the operator (105) one gets a potentially non vanishing contribution considering

$$\mathcal{O}_{4,1}^{(3)} \rightarrow \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\mathcal{D}_{\mu}^{(0)} \varphi^{(6)i} \mathcal{D}^{(0)\mu} \varphi^{(2)i} + \mathcal{D}_{\mu}^{(4)} \varphi^{(2)i} \mathcal{D}^{(0)\mu} \varphi^{(2)i} \right), \quad (113)$$

which yields the combination

$$\varepsilon_{ABCD}\varepsilon_{A'B'C'D'}\left(m_{\mathbf{f}}^{A'}m_{\mathbf{f}}^{B'}m_{\mathbf{f}}^{C'}m_{\mathbf{f}}^{D'}m_{\mathbf{f}}^{[A}m_{\mathbf{f}}^{B]}m_{\mathbf{f}}^{[C}m_{\mathbf{f}}^{D]}\right), \quad (114)$$

which can saturate the superconformal modes.

In the expansion of the two operators (106) the relevant terms are

$$\mathcal{O}_{4,1}^{(4)} \rightarrow \frac{1}{g_{\text{YM}}^3 N^{1/2}} \Sigma_{AB}^i \text{Tr} \left(\varphi^{(6)i} \lambda^{(1)\alpha A} \lambda_{\alpha}^{(1)B} + \varphi^{(2)i} \lambda^{(5)\alpha A} \lambda_{\alpha}^{(1)B} + \varphi^{(2)i} \lambda^{(1)\alpha A} \lambda_{\alpha}^{(5)B} \right) \quad (115)$$

$$\mathcal{O}_{4,1}^{(5)} \rightarrow \frac{1}{g_{\text{YM}}^3 N^{1/2}} \bar{\Sigma}_i^{AB} \text{Tr} \left(\varphi^{(2)i} \bar{\lambda}_{\dot{\alpha}A}^{(3)} \bar{\lambda}_B^{(3)\dot{\alpha}} \right), \quad (116)$$

which again can saturate the fermion integrations in a two-point function since these expressions contain respectively

$$\varepsilon_{ABCD}\varepsilon_{A'B'C'D'}\left(m_{\mathbf{f}}^{[C}m_{\mathbf{f}}^{D]}m_{\mathbf{f}}^Am_{\mathbf{f}}^Bm_{\mathbf{f}}^{A'}m_{\mathbf{f}}^{B'}m_{\mathbf{f}}^{C'}m_{\mathbf{f}}^{D'}\right) \quad (117)$$

and

$$\varepsilon_{AC'D'E'}\varepsilon_{BC''D''E''}\left(m_{\mathbf{f}}^{[A}m_{\mathbf{f}}^{B]}m_{\mathbf{f}}^{C'}m_{\mathbf{f}}^{D'}m_{\mathbf{f}}^{E'}m_{\mathbf{f}}^{C''}m_{\mathbf{f}}^{D''}m_{\mathbf{f}}^{E''}\right). \quad (118)$$

Finally the operators (107) and (108), quartic in the scalars, can contribute to two-point correlators through the terms

$$\mathcal{O}_{4,1}^{(6)} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \varphi^{(2)i} \varphi^{(2)j} \right) \quad (119)$$

$$\mathcal{O}_{4,1}^{(7)} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)i} \varphi^{(2)j} \varphi^{(2)j} \right) \quad (120)$$

$$\mathcal{O}_{4,1}^{(8)} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \right) \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \right) \quad (121)$$

$$\mathcal{O}_{4,1}^{(9)} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)i} \right) \text{Tr} \left(\varphi^{(2)j} \varphi^{(2)j} \right) \quad (122)$$

whose zero-mode structure is

$$\varepsilon_{ABCD}\varepsilon_{A'B'C'D'}\left(m_{\mathbf{f}}^{[A}m_{\mathbf{f}}^{B]}m_{\mathbf{f}}^{[C}m_{\mathbf{f}}^{D]}m_{\mathbf{f}}^{[A'}m_{\mathbf{f}}^{B']}m_{\mathbf{f}}^{[C'}m_{\mathbf{f}}^{D']}\right). \quad (123)$$

From the above analysis it is clear that the classical expressions of all the fields in the singlet sector at $\Delta_0 = 4$ can contain the correct combination of fermion zero-modes to produce non-vanishing two-point functions. The expansion of all the combinations (112), (114), (118) and (123), in which all the modes are taken to be superconformal, involves exactly two ζ 's for each flavour. Moreover at leading order in the coupling there is no dependence on the ν and $\bar{\nu}$ other than what comes from the measure. So the integration over the fermionic part of the moduli space is expected to be non-zero for all two-point correlation functions $\langle \mathcal{O}_{4,1}^{(r)}(x_1) \bar{\mathcal{O}}_{4,1}^{(s)}(x_2) \rangle$, $r, s = 1, \dots, 9$. This is a general result, in sectors in which instanton contributions are present the non-perturbative mixing should be expected to be more complicated than at the first few orders in perturbation theory. In this particular case one has to diagonalise the whole 9×9 problem to extract the instanton induced anomalous dimensions. As already remarked at least three of the

eigenvalues of the anomalous dimensions matrix should vanish, since they correspond to the operators \mathcal{C}^- , \mathcal{C}^+ and \mathcal{K}'_1 discussed above. As discussed in the introduction further simplifications arise from taking into account the constraints imposed by the PSU(2,2|4), which imply that all the operators in a multiplet have the same anomalous dimension. Therefore other operators can be eliminated from the mixing problem if they are identified as superconformal descendants of operators whose anomalous dimension is known.

We shall now compute explicitly the two-point functions involving the quartic scalar operators in (107) and (108) in the one-instanton sector ⁴. This will prove that the situation in this sector is different from what was found in sections 5.1 and 5.2.1 and indeed singlet operators at $\Delta_0 = 4$ do receive instanton corrections.

In order to evaluate the semiclassical two-point functions of these operators we need their expression in the instanton background, *i.e.* the explicit form of (119)-(122). As usual it is convenient to work with the scalar fields written as φ^{AB} . The composite operators become

$$\mathcal{O}^{(6)} = \frac{1}{g_{\text{YM}}^4 N} \varepsilon_{A_1 B_1 A_3 B_3} \varepsilon_{A_2 B_2 A_4 B_4} \text{Tr} \left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \varphi^{A_3 B_3} \varphi^{A_4 B_4} \right) \quad (124)$$

$$\mathcal{O}^{(7)} = \frac{1}{g_{\text{YM}}^4 N} \varepsilon_{A_1 B_1 A_2 B_2} \varepsilon_{A_3 B_3 A_4 B_4} \text{Tr} \left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \varphi^{A_3 B_3} \varphi^{A_4 B_4} \right) \quad (125)$$

$$\mathcal{O}^{(8)} = \frac{1}{g_{\text{YM}}^4 N} \varepsilon_{A_1 B_1 A_3 B_3} \varepsilon_{A_2 B_2 A_4 B_4} \text{Tr} \left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \right) \text{Tr} \left(\varphi^{A_3 B_3} \varphi^{A_4 B_4} \right) \quad (126)$$

$$\mathcal{O}^{(9)} = \frac{1}{g_{\text{YM}}^4 N} \varepsilon_{A_1 B_1 A_2 B_2} \varepsilon_{A_3 B_3 A_4 B_4} \text{Tr} \left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \right) \text{Tr} \left(\varphi^{A_3 B_3} \varphi^{A_4 B_4} \right). \quad (127)$$

The two combinations that are needed are thus

$$\begin{aligned} \text{Tr} \left[\left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \varphi^{A_3 B_3} \varphi^{A_4 B_4} \right) (x) \right] &= \frac{2^8 \rho^8}{[(x - x_0)^2 + \rho^2]^8} \left[\left(\zeta_{\alpha}^{B_1} \zeta^{\beta A_1} - \zeta_{\alpha}^{A_1} \zeta^{\beta B_1} \right) \right. \\ &\quad \left. \left(\zeta_{\beta}^{B_2} \zeta^{\gamma A_2} - \zeta_{\beta}^{A_2} \zeta^{\gamma B_2} \right) \left(\zeta_{\gamma}^{B_3} \zeta^{\delta A_3} - \zeta_{\gamma}^{A_3} \zeta^{\delta B_3} \right) \left(\zeta_{\delta}^{B_4} \zeta^{\alpha A_4} - \zeta_{\delta}^{A_4} \zeta^{\alpha B_4} \right) \right] (x) + \dots \end{aligned} \quad (128)$$

and

$$\begin{aligned} \text{Tr} \left[\left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \right) (x) \right] &= \frac{2^4 \rho^4}{[(x - x_0)^2 + \rho^2]^4} \\ &\quad \left[\left(\zeta_{\alpha}^{B_1} \zeta^{\beta A_1} - \zeta_{\alpha}^{A_1} \zeta^{\beta B_1} \right) \left(\zeta_{\beta}^{B_2} \zeta^{\alpha A_2} - \zeta_{\beta}^{A_2} \zeta^{\alpha B_2} \right) \right] (x) + \dots \end{aligned} \quad (129)$$

In both the previous expressions the ellipsis stands for terms involving the modes of type ν and $\bar{\nu}$ as well as terms with more than eight fermion zero modes which are not relevant in semiclassical approximation.

Substituting into the expressions for the composite fields we find that only the operators $\mathcal{O}^{(6)}$ and $\mathcal{O}^{(8)}$ can soak up the superconformal modes in the combination (41). $\mathcal{O}^{(9)}$ vanishes identically, whereas $\mathcal{O}^{(7)}$ contains terms with eight superconformal modes, but not in the required combination. We thus find that the non-vanishing correlation

⁴In the following discussion we omit the subscripts indicating the dimension and SU(4) representation.

functions in the one-instanton sector in semiclassical approximation are

$$\begin{aligned} G^{(a)}(x_1, x_2) &= \langle \mathcal{O}^{(6)}(x_1) \mathcal{O}^{(6)}(x_2) \rangle \\ G^{(b)}(x_1, x_2) &= \langle \mathcal{O}^{(8)}(x_1) \mathcal{O}^{(8)}(x_2) \rangle \\ G^{(c)}(x_1, x_2) &= \langle \mathcal{O}^{(6)}(x_1) \mathcal{O}^{(8)}(x_2) \rangle. \end{aligned} \quad (130)$$

In computing these correlation functions at leading order in the coupling the ν and $\bar{\nu}$ modes appear only in the moduli space integration measure and we find

$$\begin{aligned} \langle \mathcal{O}^{(r)}(x_1) \mathcal{O}^{(s)}(x_2) \rangle &= \frac{e^{2\pi i \tau}}{N^2(N-1)!(N-2)!} \int d\rho d^4x_0 d^5\Omega \prod_{A=1}^4 d^2\eta^A d^2\bar{\xi}^A \rho^{4N-7} \\ &\int_0^\infty dr r^{4N-3} e^{-2\rho^2 r^2} \hat{\mathcal{O}}^{(r)}(x_1; \rho, x_0; \zeta) \hat{\mathcal{O}}^{(s)}(x_2; \rho, x_0; \zeta), \end{aligned} \quad (131)$$

where as usual the classical expressions for the operators in the presence of an instanton are denoted by a hat. The exact numerical coefficients will be reinstated in the final formulae.

Since there is no dependence on ν and $\bar{\nu}$ in the integrand the integrations over the five-sphere and the radial variable r can be immediately performed and one obtains

$$\begin{aligned} \langle \mathcal{O}^{(r)}(x_1) \mathcal{O}^{(s)}(x_2) \rangle &= \frac{2^{-2N} \Gamma(2N-1) e^{2\pi i \tau}}{N^2(N-1)!(N-2)!} \\ &\int \frac{d\rho d^4x_0}{\rho^5} \prod_{A=1}^4 d^2\eta^A d^2\bar{\xi}^A \hat{\mathcal{O}}^{(r)}(x_1; \rho, x_0; \zeta) \hat{\mathcal{O}}^{(s)}(x_2; \rho, x_0; \zeta). \end{aligned} \quad (132)$$

Using (128) and (129) to compute (124) and (126) and integrating over the superconformal modes we then get

$$\begin{aligned} \langle \mathcal{O}^{(r)}(x_1) \mathcal{O}^{(s)}(x_2) \rangle &= c^{rs} \frac{3^4 5^2 \pi^{-13} 2^{-2N-15} \Gamma(2N-1) e^{2\pi i \tau}}{N^2(N-1)!(N-2)!} (x_1 - x_2)^8 \\ &\int d\rho d^4x_0 \frac{\rho^{11}}{[(x_1 - x_0)^2 + \rho^2]^8 [(x_2 - x_0)^2 + \rho^2]^8}, \end{aligned} \quad (133)$$

where the coefficients c^{rs} take into account the different prefactors in the term (41) in the expansion of the operators. By explicitly computing the contractions with the Levi-Civita tensors in the definitions of the operators we find for these coefficients

$$\begin{aligned} c^{66} &= 1 \\ c^{88} &= 2^4 \\ c^{68} &= c^{86} = -2^2. \end{aligned} \quad (134)$$

The bosonic integrals which remain to be evaluated in (133) are logarithmically divergent as can be seen introducing Feynman parameters to rewrite

$$\begin{aligned} \mathcal{J} &= \int d\rho d^4x_0 \frac{\rho^{11}}{[(x_1 - x_0)^2 + \rho^2]^8 [(x_2 - x_0)^2 + \rho^2]^8} \\ &= \frac{\Gamma(16)}{[\Gamma(8)]^2} \int_0^1 d\alpha_1 d\alpha_2 \delta(\alpha_1 + \alpha_2 - 1) \alpha_1^7 \alpha_2^7 \\ &\times \int d\rho d^4x_0 \frac{\rho^{11}}{[(x_0 - \sum_i \alpha_i x_i)^2 + \rho^2 + \alpha_1 \alpha_2 x_{12}^2]^{16}}. \end{aligned} \quad (135)$$

After the standard manipulations the ρ integral can be performed and using dimensional regularisation for the x_0 integration we get

$$\mathcal{J} = \pi^{2+\epsilon} c(4-\epsilon) \frac{\Gamma(14+\epsilon)}{[\Gamma(8)]^2} \frac{1}{(x_1-x_2)^{16-\epsilon}} \int_0^1 \frac{d\alpha}{[\alpha(1-\alpha)]^{1-\epsilon}}, \quad (136)$$

where $c(d) = 3840/[(d-20)(d-22)(d-24)(d-26)(d-28)(d-30)]$. The final integration over α produces a $1/\epsilon$ pole which is the signal of a logarithmic divergence in dimensional regularisation. In conclusion we find non-zero entries in the matrix K^{rs} defined in section 4.2 for the two-point functions (130), which can be read from (133), (134) and (136). As anticipated after equation (71), these matrix elements behave as $e^{2\pi i \tau}$ with no additional powers of g_{YM} .

The above calculation shows that $\Delta_0 = 4$ operators in the singlet do acquire an instanton induced anomalous dimension. This can be confirmed analysing the OPE of a four-point function as in section 5.2.5. In fact the t -channel, $x_{13} \rightarrow 0$, $x_{24} \rightarrow 0$, of the same four-point function (93), which is computed in appendix C, displays a singularity corresponding to the exchange of operators of dimension 4. The results of the following subsections show that this contribution to the OPE must come from operators in the singlet.

Although we have not computed all the entries in the matrix K^{rs} and diagonalised $\Gamma^{rs} = (T^{-1}K)^{rs}$, our results appear to disagree with those of [56]. There, on the basis of an OPE analysis, it was argued that only operators whose dimension remains finite in the large N limit get an instanton correction. These are multi-trace operators dual to supergravity multi-particle bound states in the AdS/CFT correspondence. We have instead shown that single trace operators, which are dual to massive string modes and whose dimension diverges in the $N \rightarrow \infty$ limit, have non-zero two-point functions in the one-instanton sector.

5.3.2 $\Delta_0 = 4$, $[1, 0, 1]$

The representation $[1, 0, 1] \equiv \mathbf{15}$ occurs in (100), in (101) and (102) with multiplicity 2 and in (103) with multiplicity 7, so there are in principle many operators to be considered. The actual number of independent gauge-invariant operators however reduces when the cyclicity of the trace is taken into account. In particular this implies that there are only single trace operators.

No operators can be constructed from (100), *i.e.* with two scalars and two covariant derivatives, because of antisymmetry.

A basis of independent operators can be obtained as follows. To realise the two operators in (101) we need to consider

$$\mathcal{O}_{4,\mathbf{15}}^{(1)[ij]} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\Sigma_{AB}^{[i} \varphi^{j]} \lambda^{\alpha A} \lambda_{\alpha}^B \right) \quad (137)$$

and

$$\mathcal{O}_{4,\mathbf{15}}^{(2)[ijkl]} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(t_{AB}^{[ijk} \varphi^{l]} \lambda^{\alpha A} \lambda_{\alpha}^B \right), \quad (138)$$

where the symbols Σ_{AB}^i are defined in (236) and $t_{AB}^{[ijk]}$ is the projector defined in (74).

Similarly from (102) we get

$$\mathcal{O}_{4,15}^{(3)[ij]} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\bar{\Sigma}_{[i}^{AB} \varphi_{j]} \bar{\lambda}_{\dot{\alpha}A} \bar{\lambda}_{\dot{\alpha}B} \right) \quad (139)$$

and

$$\mathcal{O}_{4,15}^{(4)[ijkl]} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\bar{t}_{[ijk}^{AB} \varphi_{l]} \bar{\lambda}_{\dot{\alpha}A} \bar{\lambda}_{\dot{\alpha}B} \right) . \quad (140)$$

Operators in the **15** made of four scalars can be constructed antisymmetrising a pair of indices and contracting the remaining two or completely antisymmetrising the four indices. We obtain however only one independent gauge-invariant operator,

$$\mathcal{O}_{4,15}^{(5)[ij]} = \frac{1}{g^4 N} \text{Tr} \left(\varphi^{[i} \varphi^{j]} \varphi_k \varphi^k \right) , \quad (141)$$

The other combinations of four **6**'s forming a **15** are identically zero because of cyclicity of the trace. Antisymmetrisation in (141) implies that no double trace operators can be realised.

We can now analyse the instanton contributions to two-point functions in this sector. An analysis on the lines of that in the previous subsection will show that none of the above operators can soak up the correct combination of superconformal modes to give rise to non-vanishing two-point correlators. As usual the strategy is to choose a specific component for the operators and study the zero-mode structure, the result is the same for all the components. For the operator (137) we consider for instance the component $\mathcal{O}_{4,15}^{(1)[12]}$ which is proportional to

$$\begin{aligned} & \text{Tr} \left(\varphi^{13} \lambda^{[1} \lambda^{4]} + \varphi^{13} \lambda^{[2} \lambda^{3]} + \varphi^{24} \lambda^{[1} \lambda^{4]} + \varphi^{24} \lambda^{[2} \lambda^{3]} \right. \\ & \left. - \varphi^{14} \lambda^{[1} \lambda^{3]} - \varphi^{14} \lambda^{[2} \lambda^{4]} - \varphi^{23} \lambda^{[1} \lambda^{3]} - \varphi^{23} \lambda^{[2} \lambda^{4]} \right) . \end{aligned} \quad (142)$$

In order to saturate the eight superconformal modes required to give rise to a non-zero two-point functions in all these terms the contributions one needs to consider are

$$\text{Tr} \left(\varphi^{(2)AB} \lambda^{(1)[C} \lambda^{(5)D]} + \varphi^{(2)AB} \lambda^{(5)[C} \lambda^{(1)D]} + \varphi^{(6)AB} \lambda^{(1)[C} \lambda^{(1)D]} \right) . \quad (143)$$

Expanding in this way the first term in (142) yields a contribution proportional to

$$\varepsilon_{ABCD} \zeta^1 \zeta^3 \zeta^1 \zeta^4 \zeta^A \zeta^B \zeta^C \zeta^D , \quad (144)$$

which does not contain the required combination (41), having only one mode of flavour 2. Similar considerations apply to the other terms in (142) as well as to the other components of the same operator. For the operator (139) the analysis is completely analogous and gives the same result.

For the operator (138) we can consider the component $\mathcal{O}_{4,15}^{(2)[1234]}$. After contracting the indices the result one obtains is a sum of terms of the form

$$\text{Tr} \left(\varphi^{AB} \lambda^C \lambda^C + \varphi^{AC} \lambda^B \lambda^C \right) , \quad (145)$$

which when expanded as in (143) in order to select the terms with eight fermion modes again give contributions of the type (144), where one flavour occurs thrice and one flavour

only once. The result is analogous for other components and also for the operator (140), which can be analysed much in the same way.

Similar results hold for the operator (141) constructed from only scalar fields. The [12] component of (141) is

$$\begin{aligned} \mathcal{O}_{4,15}^{(5)[12]} \sim & \text{Tr} \left\{ \left(\varphi^{14} + \varphi^{23} \right) \left(-\varphi^{13} + \varphi^{24} \right) \left[\left(\varphi^{14} + \varphi^{23} \right)^2 + \left(-\varphi^{13} + \varphi^{24} \right)^2 \right. \right. \\ & \left. \left. + \left(\varphi^{12} + \varphi^{34} \right)^2 + \left(-\varphi^{14} + \varphi^{23} \right)^2 + \left(-\varphi^{13} - \varphi^{24} \right)^2 + \left(\varphi^{12} - \varphi^{34} \right)^2 \right] \right\}. \end{aligned} \quad (146)$$

and a combination of eight fermion modes is obtained selecting in each scalar φ the term with two fermion modes, $\varphi^{(2)AB}$. It is then easy to verify that none of the terms in the resulting expansion can soak up two modes of each flavour. As in the case of the operators (137)-(140) in each term there is always a flavour occurring thrice and one occurring only once. Analogous considerations can be repeated for all the components.

In conclusion we find that none of the two-point functions $\langle \mathcal{O}_{4,15}^{(r)}(x_1) \mathcal{O}_{4,15}^{(s)}(x_2) \rangle$, $r, s = 1, \dots, 7$, in this sector receives instanton corrections, so that the corresponding non-perturbative contributions to all the anomalous dimensions vanish.

5.3.3 $\Delta_0 = 4$, $[0, 2, 0]$

Scalar operators in the $[0, 2, 0] \equiv \mathbf{20}'$ were studied in [59], where the mixing at the perturbative level was resolved at order g_{YM}^2 . In the same paper it was also argued that anomalous dimensions in this sector do not receive instanton corrections. In this section we shall re-derive this result by directly analysing the instanton contributions to two-point functions.

The basis of operators we consider is different from the starting point of [59]. Moreover as in previously considered sectors at the non-perturbative level there is mixing among all the operators at leading order in g_{YM} . From (97) which admits the decomposition (100) there is one operator that can be constructed as

$$\mathcal{O}_{4,20'}^{(1)\{ij\}} = \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\mathcal{D}_\mu \varphi^{\{i} \mathcal{D}^\mu \varphi^{j\}} \right) \equiv \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\mathcal{D}_\mu \varphi^i \mathcal{D}^\mu \varphi^j \right) - \frac{1}{6g_{\text{YM}}^2} \delta^{ij} \text{Tr} \left(\mathcal{D}_\mu \varphi^k \mathcal{D}^\mu \varphi_k \right), \quad (147)$$

where as usual $\{ij\}$ indicates symmetrisation and removal of traces in flavour space, as explicitly indicated in (147).

We then find two operators involving fermionic bilinears respectively from (101) and (102). They read

$$\mathcal{O}_{4,20'}^{(2)\{ij\}} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\Sigma_{AB}^{\{i} \varphi^{j\}} \lambda^{\alpha A} \lambda_\alpha^B \right) \quad (148)$$

and

$$\mathcal{O}_{4,20'}^{(3)\{ij\}} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\bar{\Sigma}^{AB\{i} \varphi^{j\}} \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{B}}^{\dot{\alpha}} \right). \quad (149)$$

Operators made out of four scalars in the $\mathbf{20}'$ can be single or double trace. There is a total of four operators of this type and as a basis we consider the single trace

$$\mathcal{O}_{4,20'}^{(4)\{ij\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{\{i} \varphi^{j\}} \varphi_k \varphi^k \right), \quad (150)$$

$$\mathcal{O}_{4,20'}^{(5)\{ij\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{\{i} \varphi_k \varphi^{j\}} \varphi^k \right) \quad (151)$$

and the double trace combinations

$$\mathcal{O}_{4,20'}^{(6)\{ij\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{\{i} \varphi^{j\}} \right) \text{Tr} \left(\varphi_k \varphi^k \right), \quad (152)$$

$$\mathcal{O}_{4,20'}^{(7)\{ij\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{\{i} \varphi_k \right) \text{Tr} \left(\varphi^{j\}} \varphi^k \right). \quad (153)$$

The analysis of the zero-mode structure of these operators follows closely what was done for the $[1, 0, 1]$ sector. Notice that the antisymmetrisation was irrelevant in deriving the results of the previous subsection and only played a rôle in constructing the basis of operators. Therefore considering the $\{12\}$ component of (148)-(153) we can immediately conclude that all the two-point functions $\langle \mathcal{O}_{4,20'}^{(r)\{12\}}(x_1) \mathcal{O}_{4,20'}^{(s)\{12\}}(x_2) \rangle$, for $r, s = 2, \dots, 7$, vanish in the one-instanton sector and in semiclassical approximation. The zero-mode content of (148)-(153) is in fact exactly the same found in the analogous antisymmetric combinations belonging to the **15**. If one chooses a component which is diagonal in flavour space, $\mathcal{O}_{4,20'}^{\{ii\}}$, the analysis is slightly more involved since the subtraction of the trace is then crucial in cancelling terms which could saturate the superconformal modes in a two-point function. A careful analysis of the instanton profiles of the operators shows that indeed for these components the two-point correlation functions vanish as well.

The operator (147) must be analysed separately since it has no analogue in the $[1, 0, 1]$ sector. Considering again the $\{12\}$ component the terms in the expansion which soak up eight fermion modes are

$$\begin{aligned} \mathcal{O}_{4,20'}^{(1)\{12\}} &\rightarrow \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\mathcal{D}_\mu^{(0)} \varphi^{(2)\{1} \mathcal{D}^{(4)\mu} \varphi^{(2)2\}} + \mathcal{D}_\mu^{(4)} \varphi^{(2)\{1} \mathcal{D}^{(0)\mu} \varphi^{(2)2\}} \right) . \\ &\sim \frac{1}{g_{\text{YM}}^2} \text{Tr} \left[\mathcal{D}_\mu^{(0)} \left(\varphi^{(2)14} + \varphi^{(2)23} \right) \mathcal{D}^{(4)\mu} \left(-\varphi^{(2)13} + \varphi^{(2)24} \right) + \dots \right], \end{aligned} \quad (154)$$

where the ellipsis stands for symmetrisation. Recalling that $\mathcal{D}^{(4)}$ involves one fermion mode of each flavour we find that (154) contains the terms

$$\varepsilon_{ABCD} \zeta^A \zeta^B \zeta^C \zeta^D \left(\zeta^1 \zeta^4 \zeta^1 \zeta^3 + \zeta^1 \zeta^4 \zeta^2 \zeta^4 + \zeta^2 \zeta^3 \zeta^1 \zeta^3 + \zeta^2 \zeta^3 \zeta^2 \zeta^4 \right) \quad (155)$$

none of which equals the required combination (41).

Hence we find that in the scalar $[0, 2, 0]$ sector there are no operators which acquire an anomalous dimension at the instanton level.

5.3.4 $\Delta_0 = 4$, $[2, 1, 0]$ and $[0, 1, 2]$

Operators in the representations $[2, 1, 0] \equiv \mathbf{45}$ and $[0, 1, 2] \equiv \overline{\mathbf{45}}$ arise respectively from (101) and (102), involving a scalar and a fermionic bilinear, and from (103), quartic in the scalars, which contains both **45** and $\overline{\mathbf{45}}$ with multiplicity 3. Operators of the latter type can in principle be single- or double-trace.

To project (98) onto the **45** we consider

$$\begin{aligned} \mathcal{O}_{4,45[CD]}^{(1)(AB)} = \frac{1}{g^3 N^{1/2}} \text{Tr} \left[\bar{\varphi}_{CD} \lambda^{\alpha(A} \lambda_{\alpha}^{B)} - \frac{1}{6} \left(\delta_C^A \bar{\varphi}_{ED} \lambda^{\alpha(E} \lambda_{\alpha}^{B)} \right. \right. \\ \left. \left. - \delta_D^A \bar{\varphi}_{CE} \lambda^{\alpha(E} \lambda_{\alpha}^{B)} + \delta_C^B \bar{\varphi}_{ED} \lambda^{\alpha(A} \lambda_{\alpha}^{E)} - \delta_D^B \bar{\varphi}_{CE} \lambda^{\alpha(A} \lambda_{\alpha}^{E)} \right) \right]. \end{aligned} \quad (156)$$

The operator in the $\overline{45}$ made of two fermions a scalar is obtained analogously as

$$\begin{aligned} \mathcal{O}_{4,45(AB)}^{(1)[CD]} = \frac{1}{g^3 N^{1/2}} \text{Tr} \left[\varphi^{CD} \bar{\lambda}_{\dot{\alpha}(A} \bar{\lambda}_{\dot{\alpha}}^{B)} - \frac{1}{6} \left(\delta_A^C \varphi^{ED} \bar{\lambda}_{\dot{\alpha}(E} \bar{\lambda}_{\dot{\alpha}}^{B)} \right. \right. \\ \left. \left. - \delta_A^D \varphi^{CE} \bar{\lambda}_{\dot{\alpha}(E} \bar{\lambda}_{\dot{\alpha}}^{B)} + \delta_B^C \varphi^{ED} \bar{\lambda}_{\dot{\alpha}(A} \bar{\lambda}_{\dot{\alpha}}^{E)} - \delta_B^D \varphi^{CE} \bar{\lambda}_{\dot{\alpha}(A} \bar{\lambda}_{\dot{\alpha}}^{E)} \right) \right]. \end{aligned} \quad (157)$$

Because of cyclicity of the trace from the product of four scalars we obtain only one operator in the **45** and one in the $\overline{45}$. In order to project (99) we can construct a projector onto the **45** as

$$\begin{aligned} \mathcal{P}_{[ijk]l[CD]}^{45(AB)} = \Sigma_{[i}^{AE} \bar{\Sigma}_{jEF} \Sigma_{k]}^{FB} \bar{\Sigma}_{lCD} - \frac{1}{6} \left(\delta_C^A \Sigma_{[i}^{GE} \bar{\Sigma}_{jEF} \Sigma_{k]}^{FB} \bar{\Sigma}_{lGD} - \delta_D^A \Sigma_{[i}^{GE} \bar{\Sigma}_{jEF} \Sigma_{k]}^{FB} \bar{\Sigma}_{lCG} \right. \\ \left. + \delta_C^B \Sigma_{[i}^{AE} \bar{\Sigma}_{jEF} \Sigma_{k]}^{FG} \bar{\Sigma}_{lGD} - \delta_D^B \Sigma_{[i}^{AE} \bar{\Sigma}_{jEF} \Sigma_{k]}^{FG} \bar{\Sigma}_{lCG} \right) \end{aligned} \quad (158)$$

and similarly for the $\overline{45}$ we consider

$$\begin{aligned} \mathcal{P}_{[ijk]l(AB)}^{\overline{45}[CD]} = \bar{\Sigma}_{AE[i} \Sigma_j^{EF} \bar{\Sigma}_{k]FB} \Sigma_l^{CD} - \frac{1}{6} \left(\delta_A^C \bar{\Sigma}_{GE[i} \Sigma_j^{EF} \bar{\Sigma}_{k]FB} \Sigma_l^{GD} - \delta_A^D \bar{\Sigma}_{GE[i} \Sigma_j^{EF} \bar{\Sigma}_{k]FB} \Sigma_l^{CG} \right. \\ \left. + \delta_B^C \bar{\Sigma}_{AE[i} \Sigma_j^{EF} \bar{\Sigma}_{k]FG} \Sigma_l^{GD} - \delta_B^D \bar{\Sigma}_{AE[i} \Sigma_j^{EF} \bar{\Sigma}_{k]FG} \Sigma_l^{CG} \right) \end{aligned} \quad (159)$$

Then the two operators are respectively

$$\mathcal{O}_{4,45[CD]}^{(2)(AB)} = \mathcal{P}_{[ijk]l[CD]}^{45(AB)} \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \varphi^l \right) \quad (160)$$

and

$$\mathcal{O}_{4,45(AB)}^{(2)[CD]} = \mathcal{P}_{[ijk]l(AB)}^{\overline{45}[CD]} \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \varphi^l \right). \quad (161)$$

Notice that because of the symmetry properties of the projectors (158) and (159) there is no double trace operator in these sectors.

To compute possible instanton corrections to the anomalous dimensions of the above operators we need to consider two-point functions $\langle \mathcal{O}_{4,45}(x_1) \mathcal{O}_{4,\overline{45}}(x_2) \rangle$. As usual it is convenient to pick a component and analyse the dependence on the superconformal zero-modes. A simple choice is to consider the set correlation functions

$$G^{(r,s)}(x_1, x_2) = \langle \mathcal{O}_{4,45[34]}^{(r)(12)}(x_1) \mathcal{O}_{4,\overline{45}(12)}^{(s)[34]}(x_2) \rangle, \quad r, s = 1, 2, \quad (162)$$

since for these components there is no trace to subtract in flavour space and (156), (157), (160) and (161) simplify. For the operators involving fermionic bilinears the possible non-vanishing contributions to (162) in semiclassical approximation come from the following terms

$$\mathcal{O}_{4,45[34]}^{(1)(12)} \rightarrow \frac{1}{g^3 N^{1/2}} \text{Tr} \left\{ \bar{\varphi}_{[34]}^{(2)} \left[\left(\lambda^{(1)} \right)^{\alpha(1)} \left(\lambda^{(5)} \right)_{\alpha}^{(2)} + \left(\lambda^{(5)} \right)^{\alpha(1)} \left(\lambda^{(1)} \right)_{\alpha}^{(2)} \right] \right\} \quad (163)$$

$$\mathcal{O}_{4,\overline{45}(12)}^{(1)[34]} \rightarrow \frac{1}{g^3 N^{1/2}} \text{Tr} \left[\varphi^{(2)[34]} \left(\bar{\lambda}^{(3)} \right)_{\dot{\alpha}(1)} \left(\bar{\lambda}^{(3)} \right)_{\dot{\alpha}(2)}^{\dot{\alpha}} \right]. \quad (164)$$

The combination of fermionic modes contained in (163) is

$$\mathcal{O}_{4,\mathbf{45}[34]}^{(1)(12)} \sim \varepsilon_{ABCD} m_f^1 m_f^2 m_f^1 m_f^2 m_f^A m_f^B m_f^C m_f^D \quad (165)$$

and the operator cannot provide two modes of each flavour as required. From (164) we find a contribution proportional to

$$\mathcal{O}_{4,\overline{\mathbf{45}}(12)}^{(1)[34]} \sim \varepsilon_{1ABC} \varepsilon_{2A'B'C'} m_f^3 m_f^4 m_f^A m_f^B m_f^C m_f^{A'} m_f^{B'} m_f^{C'}, \quad (166)$$

so that it cannot give rise to a non-vanishing contribution when inserted in a two-point point function since it contains only one mode of flavours 1 and 2.

For the two operators quartic in the scalars the terms in the solution which need to be considered for the semiclassical analysis are

$$\mathcal{O}_{4,\mathbf{45}[34]}^{(2)(12)} \rightarrow \mathcal{P}_{[ijk]l[34]}^{\mathbf{45}(12)} \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \varphi^{(2)k} \varphi^{(2)l} \right) \quad (167)$$

$$\mathcal{O}_{4,\overline{\mathbf{45}}(12)}^{(2)[34]} \rightarrow \mathcal{P}_{[ijk]l(12)}^{\overline{\mathbf{45}}[34]} \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^{(2)i} \varphi^{(2)j} \varphi^{(2)k} \varphi^{(2)l} \right). \quad (168)$$

Using the solution (284) and performing the contraction to project onto the $\mathbf{45}$ and $\overline{\mathbf{45}}$ one verifies that none of these two operators can saturate the superconformal modes as required for producing a non-zero two-point correlation function.

The same results can be shown for other components, in which case a cancellation involving the additional terms in the operators subtracting flavour traces is required. We conclude that the scaling dimensions of operators in these sectors do not receive instanton corrections.

5.3.5 $\Delta_0 = 4$, $[2, 0, 2]$

The representation $[2, 0, 2] \equiv \mathbf{84}$ at $\Delta_0 = 4$ can only be obtained from (103), *i.e.* the operators in this sector are all quartic in the scalars. This sector is the simplest example of the type described in [29, 40], which comprises operators transforming in representations with Dynkin labels $[a, b, a]$ with bare dimension $\Delta_0 = 2a + b$. Operators of this type cannot involve covariant derivatives, field strengths or fermions and are only made out of elementary scalars. Among these operators those which are superconformal primaries belong to 1/4 BPS multiplets [53, 57]. Moreover, as shown in [29], for fields in this class it is always possible to choose components that can be written in terms of only two complex scalars. The action of the dilation operator on composite fields belonging to these sectors is also particularly simple [29, 40].

Notice that the $[1, 1, 1]$ sector at $\Delta_0 = 3$ would also belong to this class, but, as observed in section 5.2.4, it is not realised in terms of gauge invariant operators.

The representation $\mathbf{84}$ occurs in (103) with multiplicity 2 allowing in principle to construct four independent operators, two single-trace and two double-trace operators. However cyclicity of the trace implies that there exist only two independent gauge-invariant operators. In order to obtain a single-trace operator in this representation we consider

$$\mathcal{O}_{4,\mathbf{84}}^{(1)[ij][kl]} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left\{ [\varphi^i, \varphi^j][\varphi^k, \varphi^l] - \frac{1}{4} \left(\delta^{ik} [\varphi^m, \varphi^j][\varphi^m, \varphi^l] + \delta^{il} [\varphi^m, \varphi^j][\varphi^k, \varphi^m] \right) \right\}$$

$$\begin{aligned}
& +\delta^{jk}[\varphi^i, \varphi^m][\varphi^m, \varphi^l] + \delta^{jl}[\varphi^i, \varphi^m][\varphi^k, \varphi^m]) \\
& +\frac{1}{20} \left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} \right) [\varphi^m, \varphi^n][\varphi^m, \varphi^n] \Big\} ,
\end{aligned} \tag{169}$$

i.e. we need to subtract the singlet, the **15** and the **20'** from the symmetric part of **15** \otimes **15**.

Similarly to project the double trace composite operator in (99) onto the **84** we consider

$$\begin{aligned}
\mathcal{O}_{4,\mathbf{84}}^{(2)[ij][kl]} &= \frac{1}{g_{\text{YM}}^4 N} \Big\{ \text{Tr}(\varphi^i \varphi^k) \text{Tr}(\varphi^j \varphi^l) - \text{Tr}(\varphi^i \varphi^l) \text{Tr}(\varphi^j \varphi^k) \\
& -\frac{1}{4} \left[\delta^{ik} \left(\text{Tr}(\varphi^m \varphi^m) \text{Tr}(\varphi^j \varphi^l) - \text{Tr}(\varphi^j \varphi^m) \text{Tr}(\varphi^m \varphi^l) \right) \right. \\
& +\delta^{il} \left(\text{Tr}(\varphi^m \varphi^k) \text{Tr}(\varphi^j \varphi^m) - \text{Tr}(\varphi^m \varphi^m) \text{Tr}(\varphi^j \varphi^k) \right) \\
& +\delta^{jl} \left(\text{Tr}(\varphi^i \varphi^k) \text{Tr}(\varphi^m \varphi^m) - \text{Tr}(\varphi^i \varphi^m) \text{Tr}(\varphi^m \varphi^k) \right) \\
& \left. +\delta^{jk} \left(\text{Tr}(\varphi^i \varphi^m) \text{Tr}(\varphi^m \varphi^l) - \text{Tr}(\varphi^i \varphi^l) \text{Tr}(\varphi^m \varphi^m) \right) \right] \\
& +\frac{1}{20} \left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} \right) \left[\text{Tr}(\varphi^m \varphi^m) \text{Tr}(\varphi^n \varphi^n) - \text{Tr}(\varphi^m \varphi^n) \text{Tr}(\varphi^m \varphi^n) \right] \Big\} .
\end{aligned} \tag{170}$$

Operators in this sector were studied in perturbation theory in [4, 29, 51]. The single trace operator (169) belongs to the Konishi multiplet, being a superconformal descendant of \mathcal{K}_1 at level $\delta^2 \bar{\delta}^2$. A linear combination of (170) and (169) is protected and indeed a superconformal primary and thus it is the lowest component of a 1/4 BPS multiplet. We should therefore expect to find that the instanton contributions vanish in this sector.

The computation of instanton contributions to the two-point functions $\langle \mathcal{O}_{4,\mathbf{84}}^{(r)[ij][kl]}(x_1) \mathcal{O}_{4,\mathbf{84}}^{(s)[mn][pq]}(x_2) \rangle$, $r, s = 1, 2$, at leading order in g_{YM} requires the expansion of the fields to the lowest order needed to saturate eight fermion modes at each point, so for the two above operators we need to compute

$$\mathcal{O}_{4,\mathbf{84}}^{(1)[ij][kl]} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left([\varphi^{(2)i}, \varphi^{(2)j}] [\varphi^{(2)k}, \varphi^{(2)l}] \right) + \dots \tag{171}$$

$$\begin{aligned}
\mathcal{O}_{4,\mathbf{84}}^{(2)[ij][kl]} &\rightarrow \frac{1}{g_{\text{YM}}^4 N} \left[\text{Tr}(\varphi^{(2)i} \varphi^{(2)k}) \text{Tr}(\varphi^{(2)j} \varphi^{(2)l}) \right. \\
& \left. - \text{Tr}(\varphi^{(2)i} \varphi^{(2)l}) \text{Tr}(\varphi^{(2)j} \varphi^{(2)k}) + \dots \right] .
\end{aligned} \tag{172}$$

Using the expression for $\varphi^{(2)i}$ in (284) one can verify that the expansion of the two operators defined in (169) and (170) in the instanton background does not contain the combination (41) needed to produce non-vanishing two-point functions. The simplest way of showing this is to exploit the observation of [29] and choose a component involving only two complex scalars, *e.g.*

$$\mathcal{O}_{4,\mathbf{84}}^{(1)} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \text{Tr}([\phi^1, \phi^2][\phi^1, \phi^2]) \tag{173}$$

$$\mathcal{O}_{4,\mathbf{84}}^{(2)} \rightarrow \frac{1}{g_{\text{YM}}^4 N} \left[\text{Tr}(\phi^1 \phi^2) \text{Tr}(\phi^1 \phi^2) - \text{Tr}(\phi^1 \phi^1) \text{Tr}(\phi^2 \phi^2) \right] . \tag{174}$$

Notice that with this choice no subtraction of traces is required. For both of these operators the instanton profile contains the combination

$$m_{\text{f}}^1 m_{\text{f}}^4 m_{\text{f}}^2 m_{\text{f}}^4 m_{\text{f}}^1 m_{\text{f}}^4 m_{\text{f}}^2 m_{\text{f}}^4 \quad (175)$$

of zero modes and so all their two-point functions vanish. Therefore the two operators in the **84** receive no instanton contribution to their scaling dimension as expected.

The calculation just described provides another proof (and the most direct one) of the fact that the scaling dimension of the Konishi multiplet is not corrected by instantons. As observed in section 5.1 a direct computation of two-point functions of the lowest component \mathcal{K}_1 is rather subtle. It is instead easy to analyse the superconformal descendant $\mathcal{K}_{84}^{[ij][kl]} \equiv \mathcal{O}_{4,84}^{(1)[ij][kl]}$, which has the same vanishing anomalous dimension because of supersymmetry. This situation resembles what is found in perturbation theory [29], where the action of the dilation operator is simpler on the descendant than on the primary field. This example shows explicitly a general feature, *i.e.* how the superconformal symmetry of the theory can be used to simplify the computation of anomalous dimensions.

5.3.6 $\Delta_0 = 4$, $[0, 4, 0]$

The sector of $\Delta_0 = 4$ operators transforming in the representation $[0, 4, 0] \equiv \mathbf{105}$ is in the same class as the **20'** at $\Delta_0 = 2$ and the **50** at $\Delta_0 = 3$ examined in sections 5.1 and 5.2.3 respectively. These are operators in representation with Dynkin labels $[0, \ell, 0]$ and bare dimension $\Delta_0 = \ell$ and when primaries they belong to 1/2 BPS multiplets. The dimension 4 operators of this type are selected from (99) by taking the completely symmetric and traceless combination of $\text{SU}(4)$ indices. There is a single- and a double-trace operator

$$\mathcal{O}_{4,\mathbf{105}}^{(1)\{ijkl\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \varphi^l \right) \quad (176)$$

$$\mathcal{O}_{4,\mathbf{105}}^{(2)\{ijkl\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \right) \text{Tr} \left(\varphi^k \varphi^l \right). \quad (177)$$

In this case the projection onto the **105** requires the subtraction of the **20'** and the singlet with coefficients which are readily determined imposing tracelessness.

In order to analyse the instanton contributions to these operators we consider a specific component and the simplest choice is to consider a component written in terms of a single complex scalar. In this way no trace subtraction is required and two representatives are for instance

$$\mathcal{O}_{4,\mathbf{105}}^{(1)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\phi^1 \phi^1 \phi^1 \phi^1 \right) \quad (178)$$

and

$$\mathcal{O}_{4,\mathbf{105}}^{(2)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\phi^1 \phi^1 \right) \text{Tr} \left(\phi^1 \phi^1 \right). \quad (179)$$

It is then clear that the corresponding two-point functions are not corrected at the instanton level since the operators (178) and (179) involve fermion modes in the combination $(m_{\text{f}}^1 m_{\text{f}}^4)^4$.

5.3.7 $\Delta_0 = 4, [1, 2, 1]$

The representation $[1, 2, 1] \equiv \mathbf{175}$ is contained only in the decomposition (103), where it occurs with multiplicity 3. We can thus expect single- and double-trace operators made of four scalars. However cyclicity of the trace implies that actually no gauge-invariant operator exists in this sector. This can be seen analysing the origin of the three $\mathbf{175}$'s in (103). They arise from the products $(\mathbf{15} \otimes \mathbf{15})_a$, $(\mathbf{20}' \otimes \mathbf{20}')_a$ and $\mathbf{15} \otimes \mathbf{20}'$. The first two correspond to operators of the form

$$\text{Tr} \left([\varphi^{[i} \varphi^{j]}, \varphi^{[k} \varphi^{l]}] \right) \quad (180)$$

and

$$\text{Tr} \left([\varphi^{\{i} \varphi^{j\}}, \varphi^{\{k} \varphi^{l\}}] \right) \quad (181)$$

and thus vanish as traces of commutators. For the third combination which contains the $\mathbf{175}$ we should consider

$$\text{Tr} \left(\varphi^{[i} \varphi^{j]} \varphi^{\{k} \varphi^{l\}} \right), \quad (182)$$

which again vanishes as can be shown by considering ⁵

$$\text{Tr} \left(\varphi^{[i} \varphi^{j]} \varphi^{\{k} \varphi^{l\}} \right) = \text{Tr} \left[\left(\varphi^{[i} \varphi^{j]} \varphi^{\{k} \varphi^{l\}} \right)^\dagger \right] = -\text{Tr} \left(\varphi^{[i} \varphi^{j]} \varphi^{\{k} \varphi^{l\}} \right). \quad (183)$$

The representation $\mathbf{175}$ is thus not realised in terms of gauge-invariant operators. Notice that if operators in this sector could be constructed they would be in the same class as those in the $\mathbf{84}$ discussed in section 5.3.5, *i.e.* operators transforming in a representation $[a, b, a]$ with $\Delta_0 = 2a + b$: the $\mathbf{175}$ corresponds to $a = 1, b = 2$.

5.4 Dimension 5 scalar operators

The analysis of gauge-invariant operators becomes increasingly complicated as their bare dimension grows and it is already very involved at dimension 5. The number of $\text{SU}(4)$ representations which appear and their multiplicities increase rapidly, making the construction of a basis of independent operators in each sector rather non-trivial. The discussion in this section will therefore be less detailed than in previous sections. Instanton contributions to anomalous dimensions are expected only for operators in the $\mathbf{6}$ of $\text{SU}(4)$ ⁶. Explicit calculations of two-point functions in this sector will be presented in order to display a new feature which appears, *i.e.* the necessity of including the leading quantum fluctuations around the classical instanton configuration in order to compute two-point functions at leading order in the coupling. Although rather cumbersome, the analysis can however be carried on along the lines discussed in the previous cases without additional new conceptual difficulties for all the other sectors.

The combinations of elementary fields which give rise to scalar composites of bare dimension 5 are

$$\mathcal{O}_{5,6}^{(a)i} \sim \text{Tr} \left(F_{\mu\nu} F^{\mu\nu} \varphi^i \right), \quad \mathcal{O}_{5,6}^{(b)i} \sim \text{Tr} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \varphi^i \right), \quad (184)$$

⁵I thank Massimo Bianchi for a discussion on this point.

⁶An argument supporting this claim will be presented in the next section.

$$\mathcal{O}_{5,6\otimes 6\otimes 6}^{ijk} \sim \text{Tr} \left(\varphi^i \mathcal{D}_\mu \varphi^j \mathcal{D}^\mu \varphi^k \right), \quad (185)$$

$$\mathcal{O}_{5,6\otimes 6\otimes 4\otimes 4}^{ijAB} \sim \text{Tr} \left(\varphi^i \varphi^j \lambda^{\alpha A} \lambda_\alpha^B \right), \quad \mathcal{O}_{AB;5,6\otimes 6\otimes \bar{4}\otimes \bar{4}}^{ij} \sim \text{Tr} \left(\varphi^i \varphi^j \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{B}}^{\dot{\alpha}} \right), \quad (186)$$

$$\mathcal{O}_{B;5,6\otimes 6\otimes 4\otimes \bar{4}}^{ijA} \sim \text{Tr} \left(\mathcal{D}_\mu \varphi^i \lambda^{\alpha A} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}_{\dot{B}}^{\dot{\alpha}} \right), \quad (187)$$

$$\mathcal{O}_{5,4\otimes 4}^{AB} \sim \text{Tr} \left(F_{\mu\nu} \lambda^{\alpha A} \sigma_{\alpha}^{\mu\nu\beta} \lambda_\beta^B \right), \quad \mathcal{O}_{5,4\otimes 4}^{AB} \sim \text{Tr} \left(\tilde{F}_{\mu\nu} \lambda^{\alpha A} \sigma_{\alpha}^{\mu\nu\beta} \lambda_\beta^B \right), \quad (188)$$

$$\mathcal{O}_{AB;5,\bar{4}\otimes \bar{4}} \sim \text{Tr} \left(F_{\mu\nu} \bar{\lambda}_{\dot{\alpha} A} \bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}} \bar{\lambda}_{\dot{B}}^{\dot{\beta}} \right), \quad \mathcal{O}_{AB;5,\bar{4}\otimes \bar{4}} \sim \text{Tr} \left(\tilde{F}_{\mu\nu} \bar{\lambda}_{\dot{\alpha} A} \bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}} \bar{\lambda}_{\dot{B}}^{\dot{\beta}} \right), \quad (189)$$

$$\mathcal{O}_{5,4\otimes 4}^{AB} \sim \text{Tr} \left(\mathcal{D}_\mu \lambda^{\alpha A} \mathcal{D}^\mu \lambda_\alpha^B \right), \quad \mathcal{O}_{AB;5,\bar{4}\otimes \bar{4}} \sim \text{Tr} \left(\mathcal{D}_\mu \bar{\lambda}_{\dot{\alpha} A} \mathcal{D}^\mu \bar{\lambda}_{\dot{B}}^{\dot{\alpha}} \right) \quad (190)$$

$$\mathcal{O}_{5,6\otimes 6\otimes 6\otimes 6}^{ijklm} \sim \text{Tr} \left(\varphi^i \varphi^j \varphi^k \varphi^l \varphi^m \right). \quad (191)$$

Moreover for the operators in (186) and in (191) we also need to consider the double-trace operators obtained splitting the product into two traces in all possible ways.

Operators involving the field strength only contribute to the **6** of SU(4). From the combination in (185) we obtain operators in the sectors appearing in the decomposition

$$\begin{aligned} [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] &= 3[0, 1, 0] \oplus [2, 0, 0] \oplus [0, 0, 2] \oplus [0, 3, 0] \oplus 2[1, 1, 1] \\ &\Leftrightarrow \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = 3 \cdot \mathbf{6} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{50} \oplus 2 \cdot \mathbf{64}. \end{aligned} \quad (192)$$

From (186) and the double-trace combinations involving the same fields we get operators respectively in

$$\begin{aligned} [0, 1, 0] \otimes [0, 1, 0] \otimes [1, 0, 0] \otimes [1, 0, 0] &= 4[0, 1, 0] \oplus 3[2, 0, 0] \oplus 2[0, 0, 2] \\ &\oplus [0, 3, 0] \oplus 4[1, 1, 1] \oplus [3, 0, 1] \oplus [2, 2, 0] \\ &\Leftrightarrow \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{4} \otimes \mathbf{4} = 4 \cdot \mathbf{6} \oplus 3 \cdot \mathbf{10} \oplus 2 \cdot \overline{\mathbf{10}} \oplus \mathbf{50} \oplus 4 \cdot \mathbf{64} \oplus \mathbf{70} \oplus \mathbf{126} \end{aligned} \quad (193)$$

and

$$\begin{aligned} [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 0, 1] \otimes [0, 0, 1] &= 4[0, 1, 0] \oplus 2[2, 0, 0] \oplus 3[0, 0, 2] \\ &\oplus [0, 3, 0] \oplus 4[1, 1, 1] \oplus [1, 0, 3] \oplus [0, 2, 2] \\ &\Leftrightarrow \mathbf{6} \otimes \mathbf{6} \otimes \bar{\mathbf{4}} \otimes \bar{\mathbf{4}} = 4 \cdot \mathbf{6} \oplus 2 \cdot \mathbf{10} \oplus 3 \cdot \overline{\mathbf{10}} \oplus \mathbf{50} \oplus 4 \cdot \mathbf{64} \oplus \overline{\mathbf{70}} \oplus \overline{\mathbf{126}}. \end{aligned} \quad (194)$$

The sectors which can contain operators of the form (187) are

$$\begin{aligned} [0, 1, 0] \otimes [1, 0, 0] \otimes [0, 0, 1] &= 2[0, 1, 0] \oplus [2, 0, 0] \oplus [0, 0, 2] \oplus [1, 1, 1] \\ &\Leftrightarrow \mathbf{6} \otimes \mathbf{4} \otimes \bar{\mathbf{4}} = 2 \cdot \mathbf{6} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{64}. \end{aligned} \quad (195)$$

The two types of operators in (188) and the first in (190) contribute to

$$[1, 0, 0] \otimes [1, 0, 0] = [0, 1, 0] \oplus [2, 0, 0] \Leftrightarrow \mathbf{4} \otimes \mathbf{4} = \mathbf{6} \oplus \mathbf{10} \quad (196)$$

whereas those in (188) and the second in (190) contribute to

$$[0, 0, 1] \otimes [0, 0, 1] = [0, 1, 0] \oplus [0, 0, 2] \Leftrightarrow \bar{\mathbf{4}} \otimes \bar{\mathbf{4}} = \mathbf{6} \oplus \overline{\mathbf{10}}. \quad (197)$$

Finally operators of the form (191) (and the corresponding double-trace operators) are found in the decomposition

$$\begin{aligned}
& [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] \otimes [0, 1, 0] = 16[0, 1, 0] \oplus 10([2, 0, 0] \oplus [0, 0, 2]) \\
& \oplus 10[0, 3, 0] \oplus 24[1, 1, 1] \oplus 5([3, 0, 1] \oplus [1, 0, 3]) \oplus ([2, 2, 0] \oplus [0, 2, 2]) \\
& \oplus [0, 5, 0] \oplus 5[2, 1, 2] \oplus 4[1, 3, 1] \\
& \Leftrightarrow \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = 16 \cdot \mathbf{6} \oplus 10 \cdot (\mathbf{10} \oplus \overline{\mathbf{10}}) \oplus 10 \cdot \mathbf{50} \oplus 24 \cdot \mathbf{64} \oplus 5 \cdot (\mathbf{70} \oplus \overline{\mathbf{70}}) \\
& \oplus 6 \cdot (\mathbf{126} \oplus \overline{\mathbf{126}}) \oplus \mathbf{196} \oplus 5 \cdot \mathbf{300} \oplus 4 \cdot \mathbf{384} .
\end{aligned} \tag{198}$$

From this preliminary analysis it is already apparent that many more representations appear at $\Delta_0 = 5$. Furthermore there are representations occurring with high multiplicity, which, as already observed, makes the construction of a basis of operators in the corresponding sectors rather tedious.

5.4.1 $\Delta_0 = 5$, $[0, 1, 0]$

Operators in the $[0, 1, 0] \equiv \mathbf{6}$ can be obtained from the combinations (184), (186) (which both contain the $\mathbf{6}$ with multiplicity 4), (187) (where it occurs with multiplicity 2), (190) and (191) (where it appears with multiplicity 16). Clearly these multiplicities do not take into account the cyclicity of the trace which reduces the actual number of independent gauge invariant operators. On the other hand we must consider single- and double-trace operators of the types in (186) and (191).

Two independent operators involving field strengths are

$$\mathcal{O}_{5,6}^{(1)i} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(F_{\mu\nu} F^{\mu\nu} \varphi^i \right), \quad \mathcal{O}_{5,6}^{(2)i} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \varphi^i \right). \tag{199}$$

With three scalars and two covariant derivatives we can construct the operators

$$\mathcal{O}_{5,6}^{(3)i} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\varphi^i \mathcal{D}_\mu \varphi^j \mathcal{D}^\mu \varphi_j \right), \quad \mathcal{O}_{5,6}^{(4)i} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\varphi^j \mathcal{D}_\mu \varphi^i \mathcal{D}^\mu \varphi_j \right). \tag{200}$$

Single trace operators with two scalars and two fermions are

$$\begin{aligned}
\mathcal{O}_{5,6}^{(5)i} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\Sigma_{AB}^i \varphi^j \varphi^j \lambda^{\alpha A} \lambda_\alpha^B \right), \quad \mathcal{O}_{5,6}^{(6)i} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\Sigma_{AB}^j \varphi^{[i} \varphi^{j]} \lambda^{\alpha A} \lambda_\alpha^B \right), \\
\mathcal{O}_{5,6}^{(7)i} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\Sigma_{AB}^i \varphi^{\{i} \varphi^{j\}} \lambda^{\alpha A} \lambda_\alpha^B \right), \quad \mathcal{O}_{5,6}^{(8)i} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{t}_{(AB)}^{[ij]k} \varphi^{[j} \varphi^{k]} \lambda^{\alpha A} \lambda_\alpha^B \right), \\
\mathcal{O}_{5,6}^{(9)i} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\Sigma_{AB}^j \lambda^{\alpha A} \varphi^{[i} \lambda_\alpha^B \varphi^{j]} \right)
\end{aligned} \tag{201}$$

and similarly with fermions of the opposite chirality

$$\begin{aligned}
\mathcal{O}_{5,6}^{(10)i} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{\Sigma}^{iAB} \varphi^j \varphi^j \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{\alpha} B} \right), \quad \mathcal{O}_{5,6}^{(11)i} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{\Sigma}^{jAB} \varphi^{[i} \varphi^{j]} \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{\alpha} B} \right), \\
\mathcal{O}_{5,6}^{(12)i} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{\Sigma}^{iAB} \varphi^{\{i} \varphi^{j\}} \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{\alpha} B} \right), \quad \mathcal{O}_{5,6}^{(13)i} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{t}_{[ij]k}^{(AB)} \varphi^{[j} \varphi^{k]} \bar{\lambda}_{\dot{\alpha} A} \bar{\lambda}_{\dot{\alpha} B} \right), \\
\mathcal{O}_{5,6}^{(14)i} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{\Sigma}^{jAB} \bar{\lambda}_{\dot{\alpha} A} \varphi^{[i} \bar{\lambda}_{\dot{\alpha} B} \varphi^{j]} \right).
\end{aligned} \tag{202}$$

Besides these operators there are double-trace operators obtained splitting the traces in (201) and (202) in all the ways allowed by the (anti)symmetry properties. The operators in (187) are

$$\begin{aligned}\mathcal{O}_{5,6}^{(15)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\mathcal{D}_\mu \varphi^i \lambda^{\alpha A} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}_A^{\dot{\alpha}} \right), & \mathcal{O}_{5,6}^{(16)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\lambda^{\alpha A} \mathcal{D}_\mu \varphi^i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}_A^{\dot{\alpha}} \right) \\ \mathcal{O}_{5,6}^{(17)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\bar{\Sigma}^{iC[A} \mathcal{D}_\mu \varphi_i \lambda^{\alpha B]} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}_C^{\dot{\alpha}} + \frac{1}{4} \Sigma_{AB}^i \mathcal{D}_\mu \varphi^{AB} \lambda^{\alpha C} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}_C^{\dot{\alpha}} \right).\end{aligned}\quad (203)$$

From (188) and (189) we get

$$\begin{aligned}\mathcal{O}_{5,6}^{(18)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\Sigma_{AB}^i F_{\mu\nu} \lambda^{\alpha A} \sigma^{\mu\nu\beta}_\alpha \lambda_B^{\dot{\beta}} \right), \\ \mathcal{O}_{5,6}^{(19)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\Sigma_{AB}^i \tilde{F}_{\mu\nu} \lambda^{\alpha A} \sigma^{\mu\nu\beta}_\alpha \lambda_B^{\dot{\beta}} \right), \\ \mathcal{O}_{5,6}^{(20)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\bar{\Sigma}^{iAB} F_{\mu\nu} \bar{\lambda}_{\dot{\alpha}A} \bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}} \bar{\lambda}_B^{\dot{\beta}} \right), \\ \mathcal{O}_{5,6}^{(21)i} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\bar{\Sigma}^{iAB} \tilde{F}_{\mu\nu} \bar{\lambda}_{\dot{\alpha}A} \bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}} \bar{\lambda}_B^{\dot{\beta}} \right).\end{aligned}\quad (204)$$

Operators of the form (190) actually do not contribute to the **6** because the trace of the antisymmetric combination vanishes. They are only found in the **10** and **$\bar{10}$** . Finally there are the operators made of only elementary scalars. As a set of independent gauge-invariant operators we can consider the single-trace combinations

$$\begin{aligned}\mathcal{O}_{5,6}^{(22)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^i \varphi^j \varphi^j \varphi^k \varphi^k \right), & \mathcal{O}_{5,6}^{(23)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \varphi^j \varphi^k \right), \\ \mathcal{O}_{5,6}^{(24)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \varphi^k \varphi^j \right), & \mathcal{O}_{5,6}^{(25)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \varepsilon^{ijklmn} \text{Tr} \left(\varphi^j \varphi^k \varphi^l \varphi^m \varphi^n \right)\end{aligned}\quad (205)$$

and the double-trace combinations

$$\begin{aligned}\mathcal{O}_{5,6}^{(26)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^i \varphi^j \right) \text{Tr} \left(\varphi^j \varphi^k \varphi^k \right), & \mathcal{O}_{5,6}^{(27)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^i \varphi^j \varphi^j \right) \text{Tr} \left(\varphi^k \varphi^k \right), \\ \mathcal{O}_{5,6}^{(28)i} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^i \varphi^j \varphi^k \right) \text{Tr} \left(\varphi^j \varphi^k \right).\end{aligned}\quad (206)$$

There is therefore a large number of operators in this sector. Using the same techniques employed in the previous sections it is easy to verify that all the above operators can soak up the fermion superconformal modes in the required combination (41). The terms we need to consider in the expansion of all the operators in (199)-(206) at the semi-classical level involve 10 fermion modes. Analysing the structure of the single operators we find that they all contain terms proportional to

$$\bar{\nu}^A \nu^B (\zeta^1)^2 (\zeta^2)^2 (\zeta^3)^2 (\zeta^4)^2, \quad (207)$$

so that in the computation of two-point functions,

$$G^{(r,s)}(x_1, x_2) = \langle \mathcal{O}_{5,6}^{(r)i}(x_1) \mathcal{O}_{5,6}^{(s)i}(x_2) \rangle, \quad r, s = 1, \dots, 28, \quad (208)$$

the integration over the superconformal modes is potentially non-zero. We should thus expect to find a non vanishing result for all the two-point functions in this sector, so that the mixing problem in the one-instanton sector and at leading order in g_{YM} is very involved.

Since the total number of fermion modes entering the classical expressions of the operators (199)-(206) exceeds the minimum required of eight, there is also the possibility of non-vanishing contributions to two-point functions in which pairs of fields are contracted. Ordinary semiclassical contributions to the two-point functions (208), in which all the fields are replaced by their background expression in the presence of an instanton, involve a non trivial five-sphere integration. The profiles of the operators contain a $\bar{\nu}\nu$ bilinear each leading to an integral of the form (69). The insertion of two $\bar{\nu}\nu$ bilinears from the operators induces a factor of g_{YM}^2 in the expectation value. Similarly a factor of g_{YM}^2 is produced by each propagator in our normalisations, so that the two types of contributions are of the same order and consistency requires that both are included. The counting of zero modes shows that the allowed contractions are between two φ^i 's, between a λ^A and a $\bar{\lambda}_A$ or between two vectors, A_μ . This is because in these cases the number of zero modes appearing in the correlation function is reduced by four, leaving a total of sixteen. The scalar contraction involves the propagator (285). The spinor and vector propagators, $\langle \bar{\lambda}_A(x_1) \lambda^B(x_2) \rangle$ and $\langle A_\mu(x_1) A_\nu(x_2) \rangle$, which have not been given explicitly, can be deduced from that for the scalar [47]⁷. In the two-point functions we are considering however we need the sixteen zero modes distributed in two groups of eight as in (41). This implies that fermion contractions cannot contribute because after the contraction the remaining sixteen modes are not evenly distributed. The same argument applies to vector contractions.

A complete analysis and resolution of the mixing in this sector is beyond the scope of the present paper, we shall however compute the two types of contributions for a specific two-point function of operators in (205) in order to illustrate the features and difficulties of the calculation.

Consider the correlation function

$$\begin{aligned} G(x_1, x_2) &= \langle \mathcal{O}_{5,6}^{(23)1}(x_1) \mathcal{O}_{5,6}^{(23)1}(x_2) \rangle \\ &= \frac{1}{g_{\text{YM}}^{10} N^3} \langle \text{Tr} \left[\left(\varphi^1 \varphi^i \varphi^j \varphi^i \varphi^j \right) (x_1) \right] \text{Tr} \left[\left(\varphi^1 \varphi^k \varphi^l \varphi^k \varphi^l \right) (x_2) \right] \rangle. \end{aligned} \quad (209)$$

In order to soak up the sixteen superconformal modes in the moduli space integration the relevant terms in the expansion of the two composite operators are $\text{Tr} \left(\varphi^{(2)1} \varphi^{(2)i} \varphi^{(2)j} \varphi^{(2)i} \varphi^{(2)j} \right)$. To compute the expression for the operators in (209) it is then convenient to rewrite them in terms of φ^{AB} 's as

$$\mathcal{O}_{5,6}^{(23)i} = \frac{1}{\sqrt{2} g_{\text{YM}}^5 N^{3/2}} \Sigma_{AB}^i \varepsilon_{A'B'C'D'} \varepsilon_{A''B''C''D''} \text{Tr} \left(\varphi^{AB} \varphi^{A'B'} \varphi^{A''B''} \varphi^{C'D'} \varphi^{C''D''} \right). \quad (210)$$

and use

$$\text{Tr} \left[\left(\varphi^{A_1 B_1} \varphi^{A_2 B_2} \varphi^{A_3 B_3} \varphi^{A_4 B_4} \varphi^{A_5 B_5} \right) (x) \right] = \frac{2^5 \rho^8}{[(x - x_0)^2 + \rho^2]^9}$$

⁷It should be noted that the evaluation of contributions containing vector contractions presents subtleties related to infrared divergences. See [60], chapter 4, and references therein.

$$\begin{aligned}
& \times \left(\left\{ \left[\left((\bar{\nu}^{[A_1} \nu^{A_2]}) (\zeta^{B_2} \zeta^{B_3}) (\zeta^{A_3} \zeta^{B_4}) (\zeta^{A_4} \zeta^{B_5}) (\zeta^{A_5} \zeta^{B_1}) - (A_4 \leftrightarrow B_4) - (A_5 \leftrightarrow B_5) \right. \right. \right. \\
& + (A_4 \leftrightarrow B_4, A_5 \leftrightarrow B_5) - (A_3 \leftrightarrow B_3) + (A_3 \leftrightarrow B_3, A_4 \leftrightarrow B_4) + (A_3 \leftrightarrow B_3, A_5 \leftrightarrow B_5) \\
& \left. \left. \left. - (A_3 \leftrightarrow B_3, A_4 \leftrightarrow B_4, A_5 \leftrightarrow B_5) \right) - (A_1 \leftrightarrow B_1) - (A_2 \leftrightarrow B_2) + (A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2) \right] \right. \right. \\
& + \text{“cyclic permutations”} \left. \right\} + \left\{ (\bar{\nu}^{[A_4} \nu^{B_4]}) \left[(\zeta^{A_1} \zeta^{B_2}) (\zeta^{A_2} \zeta^{B_3}) (\zeta^{A_3} \zeta^{B_5}) (\zeta^{A_5} \zeta^{B_1}) \right. \right. \\
& - (A_1 \leftrightarrow B_1) - (A_2 \leftrightarrow B_2) - (A_3 \leftrightarrow B_3) - (A_5 \leftrightarrow B_5) + (A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2) \\
& + (A_1 \leftrightarrow B_1, A_3 \leftrightarrow B_3) + (A_1 \leftrightarrow B_1, A_5 \leftrightarrow B_5) + (A_2 \leftrightarrow B_2, A_3 \leftrightarrow B_3) \\
& + (A_2 \leftrightarrow B_2, A_5 \leftrightarrow B_5) + (A_3 \leftrightarrow B_3, A_5 \leftrightarrow B_5) - (A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, A_3 \leftrightarrow B_3) \\
& - (A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, A_5 \leftrightarrow B_5) - (A_1 \leftrightarrow B_1, A_3 \leftrightarrow B_3, A_5 \leftrightarrow B_5) \\
& \left. \left. - (A_2 \leftrightarrow B_2, A_3 \leftrightarrow B_3, A_5 \leftrightarrow B_5) + (A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, A_3 \leftrightarrow B_3, A_5 \leftrightarrow B_5) \right] \right. \\
& \left. + \text{“cyclic permutations”} \right\} \Big) \tag{211}
\end{aligned}$$

which can be obtained after a lengthy but straightforward calculation utilising the solution (284) for the scalar field. In (211) only the terms relevant to the computation of (209) have been kept.

The first contribution to the two-point function (209) in semiclassical approximation, obtained substituting the background values of the composite operators, is

$$\begin{aligned}
G^{(1)}(x_1, x_2) &= \int d\mu_{\text{phys}} e^{-S_{\text{inst}}} \hat{\mathcal{O}}_{5,6}^{(23)1}(x_1; \rho, x_0; \eta, \bar{\xi}, \nu, \bar{\nu}) \hat{\mathcal{O}}_{5,6}^{(23)1}(x_2; \rho, x_0; \eta, \bar{\xi}, \nu, \bar{\nu}) \\
&= \frac{\pi^{-4N} g_{\text{YM}}^{4N-10} e^{2\pi i \tau}}{N^3 (N-1)! (N-2)!} \int d\rho d^4 x_0 \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A d^{N-2} \nu^A d^{N-2} \bar{\nu}^A \rho^{4N-13} \\
& e^{\frac{\pi^2}{16g_{\text{YM}}^2 \rho^2} \varepsilon_{ABCD} (\bar{\nu}^{[A} \nu^{B]}) (\bar{\nu}^{[C} \nu^{D]})} \frac{\rho^8}{[(x_1 - x_0)^2 + \rho^2]^9} \frac{\rho^8}{[(x_2 - x_0)^2 + \rho^2]^9} \\
& \left((\bar{\nu}^{[1} \nu^{4]}) + (\bar{\nu}^{[2} \nu^{3]}) \right)^2 \left[(\zeta^1)^2 (\zeta^2)^2 (\zeta^3)^2 (\zeta^4)^2 \right] (x_1) \left[(\zeta^1)^2 (\zeta^2)^2 (\zeta^3)^2 (\zeta^4)^2 \right] (x_2) \tag{212}
\end{aligned}$$

where only the terms giving a non-vanishing contribution have been included. An overall numerical coefficient has been omitted and will be reinstated in the final formula. The integrals over the ν and $\bar{\nu}$ modes can be performed with the aid of the generating function defined in section 2, which allows to replace them with a five-sphere integral of the type (69). The integration over the 16 superconformal modes is also easily done using (42), so that we get

$$\begin{aligned}
G^{(1)}(x_1, x_2) &= \frac{2^{-2N} (N^2 - 3N + 2) \Gamma(2N - 2) e^{2\pi i \tau}}{N^3 (N-1)! (N-2)!} (x_1 - x_2)^8 \\
& \int \frac{d^4 x_0 d\rho}{\rho^5} \frac{\rho^9}{[(x_1 - x_0)^2 + \rho^2]^9} \frac{\rho^9}{[(x_2 - x_0)^2 + \rho^2]^9}. \tag{213}
\end{aligned}$$

The second type of contribution involves one scalar contraction

$$G^{(2)}(x_1, x_2) = \frac{1}{g_{\text{YM}}^{10} N^3} \langle \text{Tr} \left[\underbrace{(\varphi^1 \varphi^i \varphi^j \varphi^i \varphi^j)}_{(x_1)} \right] \text{Tr} \left[(\varphi^1 \varphi^k \varphi^l \varphi^k \varphi^l) (x_2) \right] \rangle + \dots, \tag{214}$$

where the ellipsis stands for the other contractions according to Wick's theorem. The computation of this correlator is rather involved because of the complicated form of the scalar propagator in the instanton background. The insertion the propagator (285) eventually gives rise to two types of structures. Combining all the terms which arise from (214) we obtain (again up to an overall numerical coefficient)

$$\begin{aligned}
G^{(2)}(x_1, x_2) &= \frac{2^{-2N}\Gamma(2N-1)e^{2\pi i\tau}}{N^3(N-1)!(N-2)!} \int \frac{d^4x_0 d\rho}{\rho^5} d^5\Omega \prod_{A=1}^4 d^2\eta^A d^2\bar{\xi}^A \\
&\left[\frac{c_1}{(x_1-x_2)^2} \frac{\rho^{16}}{[(x_1-x_0)^2 + \rho^2]^8 [(x_2-x_0)^2 + \rho^2]^8} + \frac{c_2\rho^{18}}{[(x_1-x_0)^2 + \rho^2]^9 [(x_2-x_0)^2 + \rho^2]^9} \right] \\
&\left[(\zeta^1)^2(\zeta^2)^2(\zeta^3)^2(\zeta^4)^2 \right](x_1) \left[(\zeta^1)^2(\zeta^2)^2(\zeta^3)^2(\zeta^4)^2 \right](x_2) \\
&= \frac{2^{-2N}\Gamma(2N-1)e^{2\pi i\tau}}{N^3(N-1)!(N-2)!} (x_1-x_2)^8 \int \frac{d^4x_0 d\rho}{\rho^5} \left[\frac{c_2\rho^{18}}{[(x_1-x_0)^2 + \rho^2]^9 [(x_2-x_0)^2 + \rho^2]^9} \right. \\
&\quad \left. + \frac{c_1}{(x_1-x_2)^2} \frac{\rho^{16}}{[(x_1-x_0)^2 + \rho^2]^8 [(x_2-x_0)^2 + \rho^2]^8} \right]. \tag{215}
\end{aligned}$$

Summing (213) and (215) we obtain the complete result for the two-point function (209) in the one-instanton sector which reads

$$\begin{aligned}
G(x_1, x_2) &= \frac{3^2 5^2 \pi^{-15} 2^{-2N-16} e^{2\pi i\tau}}{N^3(N-1)!(N-2)!} (x_1-x_2)^8 \int d^4x_0 d\rho \\
&\left[\frac{a_1(N)\rho^{13}}{(y_1^2 + \rho^2)^9 (y_2^2 + \rho^2)^9} + \frac{1}{(x_1-x_2)^2} \frac{a_2(N)\rho^{11}}{(y_1^2 + \rho^2)^8 (y_2^2 + \rho^2)^8} \right], \tag{216}
\end{aligned}$$

where $y_i = x_i - x_0$ and all the numerical factors have been reinstated. The coefficients $a_1(N)$ and $a_2(N)$ are

$$a_1(N) = \frac{2}{3} \Gamma(2N-1), \quad a_2(N) = (N^2 - 3N + 2) \Gamma(2N-2) + \frac{2}{3} \Gamma(2N-1). \tag{217}$$

The final bosonic moduli space integrations in (216) are logarithmically divergent. The second term is exactly of the same form as that encountered in section 5.3.1. The first term in the last line of (216) is evaluated in a completely analogous way and also leads to a simple pole singularity after dimensional regularisation of the x_0 integral. As in the case of the singlets of dimension 4 we find non-zero entries in the matrix K^{rs} for operators with $\Delta_0 = 5$ transforming in the **6** of SU(4). Therefore some of the operators in this sector acquire an anomalous dimension in the one-instanton sector.

The calculation of the other two-point functions in this sector for operators made of only elementary scalars can be carried on in a similar fashion. The result is of the same form as (216) apart from the numerical coefficient. In particular the operator $\mathcal{O}_{5,6}^{(21)i}$ vanishes in a one-instanton background, so that two-point functions involving this operator only receive contribution from terms in which pairs of scalars are contracted. The double-trace operators in (206) can be treated in a similar way. The two-point functions involving the remaining operators in this sector, (199)-(203), require higher order terms in the iterative solution of the field equations for the vector and the fermions.

Without computing these two-point functions it is not possible to extract the values of the instanton induced anomalous dimensions of $\Delta_0 = 5$ operators in the **6** of SU(4). The fact that operators in this sector do receive instanton corrections is however also confirmed by the analysis of the four-point function in appendix C. The singularity observed in the OPE studied there corresponds to the contribution of operators in the sector examined in this subsection.

5.4.2 $\Delta_0 = 5$, $[2, 0, 0]$ and $[0, 0, 2]$

In this and the following subsections we briefly discuss the other SU(4) representations which appear at the level $\Delta_0 = 5$. The discussion will not be detailed. The analysis follows in a straightforward way what was done in previous sections. What makes the study of these sectors more involved is the large number of operators which makes the construction of a basis rather laborious. In particular we shall only consider single-trace operators. It is understood that for all operators involving at least four elementary fields there exist also double-trace operators. From the discussion in the previous sections it is evident that for a qualitative analysis of the zero-mode structure the number of traces is not relevant.

Operators in the representation $[2, 0, 0] \equiv \mathbf{10}$ are obtained from (185)-(191). The combination (185) can be projected onto the **10** by fully antisymmetrising the indices on the scalars

$$\mathcal{O}_{5,\mathbf{10}}^{(1)[ijk]} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\mathcal{D}^\mu \varphi^{[i} \mathcal{D}_\mu \varphi^j \varphi^{k]} \right). \quad (218)$$

Operators in the **10** containing fermions come from (186)-(190). We find

$$\begin{aligned} \mathcal{O}_{5,\mathbf{10}}^{(2)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \varphi^j \lambda^{\alpha(A} \lambda_{\alpha}^{B)} \right), \quad \mathcal{O}_{5,\mathbf{10}}^{(3)(AB)} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\varphi^i \lambda^{\alpha(A} \varphi_i \lambda_{\alpha}^{B)} \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(4)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} t_{[ijk]}^{(AB)} \text{Tr} \left(\Sigma_{CD}^i \varphi^j \varphi^k \lambda^{\alpha C} \lambda_{\alpha}^D \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(5)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left[\varphi^i \varphi^j \left(\Sigma_{ijC}^A \lambda^{\alpha(B} \lambda_{\alpha}^{C)} + \Sigma_{ijC}^B \lambda^{\alpha(A} \lambda_{\alpha}^{C)} \right) \right], \\ \mathcal{O}_{5,\mathbf{10}}^{(6)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\left(\Sigma_{ijC}^A \varphi^i \lambda^{\alpha(B} \varphi^j \lambda_{\alpha}^{C)} + \Sigma_{ijC}^B \varphi^i \lambda^{\alpha(A} \varphi^j \lambda_{\alpha}^{C)} \right) \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(7)(AB)} &= \frac{1}{g_{\text{YM}}^2} \text{Tr} \left(\mathcal{D}_\mu \lambda^{\alpha(A} \mathcal{D}^\mu \lambda_{\alpha}^{B)} \right), \quad \mathcal{O}_{5,\mathbf{10}}^{(8)} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} t_{[ijk]}^{(AB)} \text{Tr} \left(\Sigma_C^{ijD} \mathcal{D}_\mu \varphi^k \lambda^{\alpha C} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}_{\dot{\alpha}D} \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(9)(AB)} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(F_{\mu\nu} \lambda^{\alpha(A} \sigma_{\alpha}^{\mu\nu\beta} \lambda_{\beta}^{B)} \right), \quad \mathcal{O}_{5,\mathbf{10}}^{(10)(AB)} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\tilde{F}_{\mu\nu} \lambda^{\alpha(A} \sigma_{\alpha}^{\mu\nu\beta} \lambda_{\beta}^{B)} \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(11)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} t_{[ijk]}^{(AB)} \text{Tr} \left(\bar{\Sigma}^{iCD} \varphi^j \varphi^k \bar{\lambda}_{\dot{\alpha}C} \bar{\lambda}_{\dot{\alpha}D} \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(12)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} t_{[ijk]}^{(AB)} \text{Tr} \left(\bar{\Sigma}^{iCD} \varphi^j \bar{\lambda}_{\dot{\alpha}C} \varphi^k \bar{\lambda}_{\dot{\alpha}D} \right), \\ \mathcal{O}_{5,\mathbf{10}}^{(13)(AB)} &= \frac{1}{g_{\text{YM}}^4 N} \bar{\Sigma}_i^{C(A} \bar{\Sigma}_j^{B)D} \text{Tr} \left(\varphi^{(i} \varphi^{j)} \bar{\lambda}_{\dot{\alpha}C} \bar{\lambda}_{\dot{\alpha}D} \right), \end{aligned}$$

$$\mathcal{O}_{5,10}^{(14)(AB)} = \frac{1}{g_{\text{YM}}^4 N} \bar{\Sigma}_i^{C(A} \bar{\Sigma}_j^{B)D} \text{Tr} \left(\varphi^{(i} \bar{\lambda}_{\dot{\alpha}C} \varphi^j) \bar{\lambda}_{\dot{\alpha}D} \right). \quad (219)$$

With only elementary scalars we can construct the following combinations in the **10**

$$\begin{aligned} \mathcal{O}_{5,10}^{(15)(AB)} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} t_{[ijk]}^{(AB)} \text{Tr} \left(\varphi^l \varphi^l \varphi^i \varphi^j \varphi^k \right), \\ \mathcal{O}_{5,10}^{(16)(AB)} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} t_{[ijk]}^{(AB)} \text{Tr} \left(\varphi^l \varphi^i \varphi^l \varphi^j \varphi^k \right). \end{aligned} \quad (220)$$

The operators in the representation $[0, 0, 2] \equiv \overline{\mathbf{10}}$ are built in a completely analogous way. Moreover for operators made either of two scalars and two fermions or of five scalars there are also double-trace combinations. We should then consider the two-point functions $\langle \mathcal{O}_{5,10}^{(r)}(x_1) \bar{\mathcal{O}}_{5,\overline{\mathbf{10}}}^{(s)}(x_2) \rangle$ to compute the instanton corrections to the matrix of anomalous dimensions. By choosing specific components one can verify that for all the above operators and their conjugates the instanton contributions to two-point functions vanish. There are always 10 fermion modes involved and depending on the choice of components it may be possible to soak up the sixteen superconformal modes, but when this is the case the remaining integral over the ν and $\bar{\nu}$ fermion variables vanishes. There is no instanton correction to the scaling dimension of operators in these sectors.

5.4.3 $\Delta_0 = 5, [0, 3, 0]$

The construction of operators in the representation $[0, 3, 0] \equiv \mathbf{50}$ is very similar to what was done in the previous subsection for the **10**: in the product of three **6**'s the **50** is obtained from the totally symmetric and traceless combination and the **10** arises as totally antisymmetric combination.

We find the following operators in this sector. From (185) we select the **50** taking

$$\mathcal{O}_{5,50}^{(1)\{ijk\}} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\varphi^{\{i} \mathcal{D}_\mu \varphi^j \mathcal{D}^\mu \varphi^k\} \right). \quad (221)$$

With two scalars and two fermions we get

$$\mathcal{O}_{5,50}^{(2)\{ijk\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\Sigma_{AB}^{\{i} \varphi^j \varphi^k\} \lambda^{\alpha A} \lambda_\alpha^B \right), \quad \mathcal{O}_{5,50}^{(3)\{ijk\}} = \frac{1}{g_{\text{YM}}^4 N} \text{Tr} \left(\bar{\Sigma}^{AB\{i} \varphi^j \varphi^k\} \bar{\lambda}_{\dot{\alpha}A} \bar{\lambda}_{\dot{\alpha}B} \right). \quad (222)$$

Finally there are the operators made of scalars only

$$\begin{aligned} \mathcal{O}_{5,50}^{(4)\{ijk\}} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^l \varphi_l \varphi^{\{i} \varphi^j \varphi^k\} \right), \\ \mathcal{O}_{5,50}^{(5)\{ijk\}} &= \frac{1}{g_{\text{YM}}^5 N^{3/2}} \text{Tr} \left(\varphi^l \varphi^{\{i} \varphi_l \varphi^j \varphi^k\} \right). \end{aligned} \quad (223)$$

As in the previous case one can easily verify that none of the above operators gives rise to non-vanishing two-point functions in the instanton background. Again depending on the component considered the vanishing of the two-point functions follows from the five-sphere integration after re-expressing the dependence on the ν and $\bar{\nu}$ modes in terms of angles Ω^{AB} .

5.4.4 $\Delta_0 = 5$, $[1, 1, 1]$, $[3, 0, 1]$, $[1, 0, 3]$, $[2, 2, 0]$ and $[0, 2, 2]$

The representations $[1, 1, 1] \equiv \mathbf{64}$, $[3, 0, 1] \equiv \mathbf{70}$, $[1, 0, 3] \equiv \overline{\mathbf{70}}$, $[2, 2, 0] \equiv \mathbf{126}$ and $[0, 2, 2] \equiv \overline{\mathbf{126}}$ are rather complicated to analyse. They appear with high multiplicity and to disentangle them one needs projectors which are not straightforward to construct. We shall only briefly sketch how one can proceed to define a basis of operators. In section 6 an argument will be given that implies that there cannot be instanton contributions to two-point functions of operators in these sectors.

The $\mathbf{64}$ appears in (185), (186), (187) and (191). The first type of operator however does not contribute for the same reason discussed in section 5.2.4. The operators with fermions come from

$$\begin{aligned} \mathbf{15} \otimes \mathbf{6} &\rightarrow \text{Tr} \left(\Sigma_{AB}^i \varphi^{[j} \varphi^{k]} \lambda^{\alpha A} \lambda_{\alpha}^B \right) \Big|_{\mathbf{64}}, & \mathbf{20}' \otimes \mathbf{6} &\rightarrow \text{Tr} \left(\Sigma_{AB}^i \varphi^{\{j} \varphi^{k\}} \lambda^{\alpha A} \lambda_{\alpha}^B \right) \Big|_{\mathbf{64}}, \\ \mathbf{15} \otimes \mathbf{10} &\rightarrow \text{Tr} \left(\varphi^{[i} \varphi^{j]} \lambda^{\alpha(A} \lambda_{\alpha}^{B)} \right) \Big|_{\mathbf{64}}, & \mathbf{20}' \otimes \mathbf{10} &\rightarrow \text{Tr} \left(\varphi^{\{i} \varphi^{j\}} \lambda^{\alpha(A} \lambda_{\alpha}^{B)} \right) \Big|_{\mathbf{64}}, \\ \mathbf{15} \otimes \mathbf{6} &\rightarrow \text{Tr} \left[\mathcal{D}_{\mu} \varphi^i \left(\lambda^{\alpha A} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\lambda}_{\dot{B}}^{\alpha} - \frac{1}{4} \delta^A_B \lambda^{\alpha C} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\lambda}_{\dot{C}}^{\alpha} \right) \right] \Big|_{\mathbf{64}}, \end{aligned} \quad (224)$$

where the notation indicates that each combination has to be suitably projected onto the $\mathbf{64}$, which requires to make it orthogonal to the operators in the other representations appearing in the same tensor product. The operators involving spinors in the $\overline{\mathbf{4}}$ can be obtained in a similar fashion. The construction of operators in the $\mathbf{64}$ made of five scalars is very involved. The $\mathbf{64}$ appears in $\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6}$ with multiplicity 24. This does not take into account the cyclicity of the trace. To incorporate this we can build the operators leaving one scalar fixed in the first position in the trace and combining the remaining four scalars respectively into the representations $\mathbf{15}$, $\mathbf{20}'$, $\mathbf{45}$, $\overline{\mathbf{45}}$, $\mathbf{84}$ and $\mathbf{175}$. The corresponding combinations have been discussed in the section on operators of bare dimension $\Delta_0 = 4$. We then need to project the tensor products $\mathbf{6} \otimes \mathbf{15}$, $\mathbf{6} \otimes \mathbf{20}'$, $\mathbf{6} \otimes \mathbf{45}$, $\mathbf{6} \otimes \overline{\mathbf{45}}$, $\mathbf{6} \otimes \mathbf{84}$ and $\mathbf{6} \otimes \mathbf{175}$, which all include the $\mathbf{64}$.

The $\mathbf{70}$ appears in (186) and (191). The operator made of two scalars and two fermions is contained in the first combination in the second line of (224). The operators made of only scalars can be extracted with the procedure outlined in the previous paragraph from $\mathbf{6} \otimes \mathbf{45}$ and $\mathbf{6} \otimes \mathbf{84}$. For the $\overline{\mathbf{70}}$ one can proceed in a similar way.

Operators in the $\mathbf{126}$ also arise from (186) and (191). One operator of the form (186) is in the decomposition of the second combination in the second line of (224). Then there are operators made of five scalars which can be obtained from $\mathbf{6} \otimes \mathbf{45}$ and $\mathbf{6} \otimes \mathbf{175}$. Again the conjugate operators in the $\overline{\mathbf{126}}$ are treated in a completely analogous way.

The same reasoning can be applied to the construction of double-trace operators in these sectors.

5.4.5 $\Delta_0 = 5$, $[0, 5, 0]$

In the representation $[0, 5, 0] \equiv \mathbf{196}$ we find one single-trace and one double-trace operator. These only involve elementary scalars and belong to the same class as the operators discussed in sections 5.1, 5.2.3 and 5.3.6, *i.e.* the class of scalar operators of dimension $\Delta_0 = \ell$ transforming in the representation $[0, \ell, 0]$. For such operators it is always possible

to choose a component which can be written in terms of only one complex scalar, ϕ^I . For the single-trace operator in the **196** we can take $\text{Tr}(\phi^1\phi^1\phi^1\phi^1\phi^1)$, which is immediately verified to have vanishing two-point functions in the instanton background. This is in fact a 1/2 BPS operator dual to a third Kaluza–Klein excited mode of a supergravity scalar in $\text{AdS}_5 \times S^5$. A similar argument can be made for the double-trace operator.

5.4.6 $\Delta_0 = 5$, $[2, 1, 2]$ and $[1, 3, 1]$

Operators in the representations $[2, 1, 2] \equiv \mathbf{300}$ and $[1, 3, 1] \equiv \mathbf{384}$ can only be obtained from the product of five scalars, *i.e.* from (191) and the analogous double-trace combinations. Taking into account the cyclicity of the trace one finds that the representation **384** is not realised in terms of gauge-invariant operators. This is analogous to what was observed for the **64** at $\Delta_0 = 3$ and the **175** at $\Delta_0 = 4$. Operators in the **300** belong to the class [29] of operators of dimension $\Delta_0 = 2a + b$ transforming in the representation $[a, b, a]$, for which it is always possible to single out components that only involve two complex scalars fields, ϕ^I and ϕ^J . From equations (239) in appendix A and the discussion in the previous sections it is then clear that operators of this type cannot contain the required combination (41) of superconformal fermion zero-modes and therefore all their two-point functions vanish in the instanton background. This applies to both single- and double-trace operators.

6 Discussion and conclusions

In the previous sections we have analysed instanton contributions to two-point correlation functions of scalar operators of bare dimension $\Delta_0 = 2, 3, 4, 5$ in the $\mathcal{N}=4$ SYM theory in the semiclassical approximation. In this way it has been possible to identify the sectors in which operators get instanton corrections to their scaling dimensions. One of our motivations was to try to understand to what extent the S-duality of the theory manifests itself in the spectrum of anomalous dimensions at weak coupling. It is somewhat surprising that very few operators among those considered are corrected by instantons. Our results show that there is a large class of operators that display non-renormalisation properties in topologically non-trivial sectors. It was already known [38] that operators belonging to the Konishi multiplet, which acquire an anomalous dimension in perturbation theory, are not corrected by instanton effects. We have now shown that the same is true for the majority of the scalar operators of bare dimension $\Delta_0 \leq 5$. As already remarked these results imply the absence of instanton corrections to a much larger set of operators. The non-renormalisation properties extend to all the components of the multiplets for which a representative was found not to be corrected. Many of the operators that have been studied in this paper, and for which no instanton corrections arise, have also been studied in perturbation theory, as well as at strong coupling via the dual supergravity amplitudes in $\text{AdS}_5 \times S^5$ [29, 40, 51, 56, 58, 59]. At the perturbative level the scaling dimensions of those belonging to long multiplets do get an anomalous quantum correction. The non-renormalisation properties observed in this paper are therefore rather surprising, in particular in view of the S-duality of the theory. Although single operators might transform in a complicated way under S-duality the full spectrum of scaling dimensions

must be invariant and instantons are expected to play a crucial rôle in implementing the duality as is the case in the dual type IIB string theory.

A more careful analysis shows however that the observed non-renormalisation properties are specific to the cases of operators of relatively small bare dimension considered here. In order to explain this we need to recall the general discussion of instanton contributions to two-point functions in section 4.1. In the one-instanton sector (and up to a non-zero overall numerical constant) a generic two-point function is given by

$$\langle \bar{\mathcal{O}}^r(x_1) \mathcal{O}^s(x_2) \rangle = c^{rs} \alpha(m, n; N) g_{\text{YM}}^{8+n+m} e^{2\pi i \tau} \int d\rho d^4 x_0 d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \rho^{n+m-5} \hat{\mathcal{O}}^r(x_1; x_0, \rho, \eta, \bar{\xi}, \bar{\nu}\nu(\Omega)) \hat{\mathcal{O}}^s(x_2; x_0, \rho, \eta, \bar{\xi}, \bar{\nu}\nu(\Omega)) , \quad (225)$$

where n and m denote respectively the number of $(\bar{\nu}\nu)_6$ and $(\bar{\nu}\nu)_{10}$ bilinears entering the expressions for the operators in the instanton background. The notation in the second line indicates that these bilinears have been re-expressed in terms of the angular variables Ω^{AB} . The N -dependence is contained in the function $\alpha(m, n; N)$ defined in equation (21). In evaluating (225) one first computes the integrations over the superconformal modes which yield a non-vanishing result if and only if both operators contain the combination $(\zeta^1 \bar{\zeta}^1)(\zeta^2 \bar{\zeta}^2)(\zeta^3 \bar{\zeta}^3)(\zeta^4 \bar{\zeta}^4)$. After performing the integrals over the η 's and $\bar{\xi}$'s one is left with the integration over the five-sphere and over the original bosonic collective coordinates, x_0 and ρ . The latter integrals are in general logarithmically divergent as follows from dimensional analysis, signalling a contribution to the matrix of anomalous dimensions.

Now consider the cases of the operators studied in this paper. The profiles of operators of dimension $\Delta_0 = 2$ and 4 contain no dependence on Ω^{AB} , those of operators of dimension $\Delta_0 = 3$ and 5 are linear in Ω^{AB} . Let us assume that there are operators with $\Delta_0 = 2, 4$ in two different sectors, *i.e.* transforming in different (and not conjugate) representations \mathbf{r}_1 and \mathbf{r}_2 , which both receive instanton corrections, so that the two-point functions $\langle \mathcal{O}_{\Delta_0, \mathbf{r}_i}(x_1) \mathcal{O}_{\Delta_0, \mathbf{r}_i}(x_2) \rangle_{\text{inst}}$, $i = 1, 2$, are different from zero. This would lead to a paradox. Since the five-sphere integral in this case is trivial we would find that the two-point function $\langle \mathcal{O}_{\Delta_0, \mathbf{r}_1}(x_1) \mathcal{O}_{\Delta_0, \mathbf{r}_2}(x_2) \rangle_{\text{inst}}$ is also non-zero, but this is forbidden by the SU(4) symmetry. The same situation would arise if there were instanton corrections to two-point functions of both an operator of dimension 2 and one of dimension 4. In this case we would have an even worse situation: there would be a non-vanishing two-point function in which the two operators have different dimension and this would violate conformal invariance. A similar argument can be repeated in the cases $\Delta_0 = 3$ and 5. In these cases the five-sphere integrals are of the form

$$\int d^5 \Omega \Omega^{AB} \Omega^{CD} = \frac{1}{4} \varepsilon_{ABCD} . \quad (226)$$

Again we would find that if there were non-zero two-point functions in two different sectors, then the mixed two-point functions would also involve the same integral (226) and so would not vanish, thus violating the SU(4) symmetry. There is no similar problem associated with overlapping of different SU(4) representations for two-point functions of one operator with $\Delta_0 = 3, 5$ and one with $\Delta_0 = 2, 4$. In this case we would get a

five-sphere integral of the form

$$\int d^5\Omega \Omega^{AB}, \quad (227)$$

which vanishes identically.

This argument shows that there can be instanton corrections to the anomalous dimensions in one and only one sector among those present at $\Delta_0 = 2, 4$ and in one and only one sector among those with $\Delta_0 = 3, 5$. This is consistent with our findings in the previous sections, where by direct inspection we have shown that instantons only correct operators in the sectors $(\Delta_0 = 4, [0, 0, 0])$ and $(\Delta_0 = 5, [0, 1, 0])$. What is special about the cases we have considered is that they always involve (almost) trivial five-sphere integrals. The general situation, expected for operators of larger dimension, is that a more complicated dependence on the angles Ω^{AB} should enter in the operator profiles. In this way it is possible to have non-vanishing contributions in various sectors, but zero overlap between different sectors since the angular dependence makes them ‘orthogonal’ with respect to the five-sphere integration

$$\int d^5\Omega \tilde{\mathcal{O}}_{\mathbf{r}_i}(x_1; x_0, \rho, \Omega) \tilde{\mathcal{O}}_{\mathbf{r}_j}(x_2; x_0, \rho, \Omega) \neq 0 \quad \text{iff} \quad \mathbf{r}_j = \bar{\mathbf{r}}_i, \quad (228)$$

where we have indicated with a tilde the expressions of the operators after the integration over the superconformal zero-modes. It is therefore natural to conjecture that the operators analysed here are special and that generically for larger values of Δ_0 most of the operators that receive corrections in perturbation theory are also corrected by instantons.

Some of the non-renormalisation properties are not restricted to small values of the bare dimension. All the operators in the $SU(2)$ subsector transforming in a representation $[a, b, a]$ with $\Delta_0 = 2a + b$, as well as the other components of multiplets containing such operators, do not receive instanton corrections, irrespective of the value of Δ_0 .

The above argument explains why so many operators among those considered appear to be ‘protected’ against instanton effects, whereas they get corrections in perturbation theory. On the other hand comparing the perturbative and non-perturbative sectors we notice that operator mixing is much more complicated at the instanton level. At small orders in perturbation theory it is possible to identify subsets of operators that do not mix with others in the same sector and thus compute the anomalous dimensions within the subsector. For instance at one loop two-point functions involving one operator made of only scalars and one containing fermion bilinears in the same sector vanish. For this reason in [29, 58] it was possible to compute the one-loop anomalous dimension of the $SU(4)$ singlet scalar operators (107)-(108) without having to consider the mixing with other operators in that sector. The study of instanton corrections to two-point functions shows that in general whenever a sector receives correction the mixing occurs among all the operators in that sector. This phenomenon was to be expected: in general the mixing begins to be relevant at some order in perturbation theory and the instanton effects we have considered are subleading with respect to contributions at any order in the perturbative expansion.

In the present paper we have restricted our attention to the one-instanton sector. Part of the results remain valid in multi-instanton sectors. In particular the non-renormalisation results hold for arbitrary instanton number K , since they only rely on the analysis of the superconformal zero-modes. The calculation in sectors in which there

are non-vanishing contributions is in general much more complicated for $K > 1$. Some cases however can be treated for any K in the large N limit. This is true for sectors in which at leading order in g_{YM} no non-exact modes enter in the operators and contractions are also not allowed. This is the case in particular for the singlet at $\Delta_0 = 4$: the two-point functions for arbitrary K in the large N limit coincide with the one-instanton results of section 5.3.1 up to a calculable K -dependent coefficient. This is because a saddle-point approximation can be used for large N . When there is a non-trivial dependence on the non-exact modes their evaluation at the saddle-point is complicated and the generalisation much more difficult.

The extension of the calculations presented here to the case of orthogonal or symplectic gauge groups is also straightforward. It should be noted however that the construction of bases of independent gauge-invariant operators is significantly different in these cases. We have considered only Lorentz scalars, but the same methods can be used to study non-scalar operators. In this case the full set of PSU(2,2|4) quantum numbers, $(\Delta_0, J_1, J_2; [a, b, c])$ is needed to identify the various sectors. Therefore the spectrum is richer and the construction of independent operators in each sector more involved, but once this is done the computation of anomalous dimensions proceeds on the same lines as in the cases considered in this paper.

The non-renormalisation properties that we have derived in semiclassical approximation can be argued to remain valid at higher orders in g_{YM} . The simplest contributions beyond the semiclassical result come from additional insertions of $\bar{\nu}\nu$ bilinears, which modify the integrations over the angular variables Ω^{AB} . The resulting five-sphere integrals are only non-vanishing if an equal number of modes of each flavour is contained in the combination of $\bar{\nu}\nu$'s. This implies that including subleading terms with more fermion modes cannot produce non-zero results for two-point functions which vanish at leading order. A similar argument can be made for higher-order corrections involving contractions because the propagator (285) is proportional to ε^{ABCD} .

As previously observed non-protected operators of large dimension are expected generically to also receive instanton corrections. It is therefore natural to ask what the situation is for the operators which are relevant for the BMN limit. These are operators of large dimension and large charge with respect to one of the generators of the R-symmetry group and the techniques developed here can be applied to the study of such operators. Instanton effects in the BMN limit and the comparison with string theory in a plane wave background are currently under investigation [61].

As discussed in the introduction, recently there have been many interesting developments in connection with the computation of anomalous dimensions in the $\mathcal{N}=4$ SYM theory, leading to new non-trivial tests of the AdS/CFT correspondence [2, 7, 10–12, 18, 19, 23, 25]. Of particular interest is the emergence of an integrability structure in the theory [27, 29, 30, 41], which appears to have a counterpart in the dual string theory. On the string side this integrability property appears at the level of the semiclassical analysis of solitonic string configurations [19, 24] as well as through the emergence of an infinite set of non-local classically conserved charges [32, 33]. In the gauge theory the integrability arises most naturally when the problem of computing scaling dimensions for gauge-invariant operators is reformulated in terms of an eigenvalue problem for the

dilation operator, \hat{D} , [29, 30, 41]. \hat{D} acts on gauge-invariant composite operators as

$$\hat{D}\mathcal{O}^{(r)} = \Delta_r \mathcal{O}^{(r)} \quad (229)$$

where Δ_r is the scaling dimension of $\mathcal{O}^{(r)}$. Therefore knowing the form of the dilation operator allows to compute anomalous dimensions in a very efficient way expanding (229). This is the approach followed in [29, 40, 41] at the perturbative level.

The results of the present paper do not seem to be relevant for the issue of the integrability of $\mathcal{N}=4$ SYM. This is because if indeed an integrable structure survives in the full quantum theory it is expected to arise only in the planar limit. In this limit instanton effects are exponentially suppressed, since they produce contributions of order $e^{-8\pi^2/g_{\text{YM}}^2} \sim e^{-8\pi^2 N/\lambda}$. On the other hand the possibility of computing instanton induced corrections to the dilation operator in $\mathcal{N}=4$ SYM is very interesting in itself and our results represent a first step in this direction. Although we have not addressed this issue in the previous sections, we can make some general considerations based on the results obtained for two-point functions. First of all in the closed $\text{SU}(2)$ sector of operators of dimension $\Delta_0 = 2a + b$ transforming in the $[a, b, a]$ of $\text{SU}(4)$ the dilation operator does not receive non-perturbative corrections at all. The same is true for the even simpler sector of $\Delta_0 = \ell$ operators in the $[0, \ell, 0]$. This has been verified explicitly in sections 5.1 ($\Delta_0 = 2$, $[0, 2, 0]$), 5.2.3 ($\Delta_0 = 3$, $[0, 3, 0]$), 5.3.5 ($\Delta_0 = 4$, $[2, 0, 2]$), 5.3.6 ($\Delta_0 = 4$, $[0, 4, 0]$), 5.4.5 ($\Delta_0 = 5$, $[0, 5, 0]$) and 5.4.6 ($\Delta_0 = 5$, $[2, 1, 2]$ and $[1, 3, 1]$). Using the fact that operators in these sectors have components that can be written in terms of only two complex scalars it is possible to generalise the result to arbitrary subsectors of the above type.

In general however the dilation operator contains instanton corrections as well and it can be expanded as

$$\hat{D} = \sum_{n=0}^{\infty} c_n(N) g_{\text{YM}}^n \hat{D}_n + \sum_{K>0} \sum_{m=0}^{\infty} c_{(K,m)}(N) g_{\text{YM}}^m e^{2\pi i K \tau} \hat{D}_{(K,m)}, \quad (230)$$

where the first sum denotes the perturbative contributions and the second double sum incorporates the instanton corrections including the perturbative fluctuations around the leading semiclassical term in each instanton sector. An analogous series of anti-instanton contributions (proportional to $e^{2\pi i \bar{\tau} K}$) has not been indicated explicitly.

Focusing on the non-perturbative part, it is natural to construct the terms $\hat{D}_{(K,m)}$ not as operators acting on the elementary fields of the $\mathcal{N}=4$ theory, as in perturbation theory, but as operators acting on the multi-instanton collective coordinates. In other words one can consider the action of the dilation operator as realised on the instanton moduli space. In [62] it was shown that the supersymmetry algebra (in the $\mathcal{N}=2$ case) can be realised in a simple and elegant way on the ADHM collective coordinates (see appendix B) before imposing the ADHM constraints. The construction of the instanton supermultiplet acting with broken supersymmetries on the bosonic solution, which was outlined in section 3 for the $\text{SU}(2)$ case, is an example of application of this idea. As already observed, this approach can be implemented directly in superspace, at least for $\mathcal{N}=1$ supersymmetric theories. In [63] a superspace description of the $\text{SU}(2)$ one-instanton moduli space in $\mathcal{N}=4$ SYM was presented and the corresponding superconformal transformations were used in the computation of instanton contributions to Wilson loops. The realisation of

symmetries on the instanton moduli space can be generalised to the whole superconformal group and in particular the action of dilations can be analysed. The strategy of [29, 40] was to write down the most general form of \tilde{D} compatible with the structure of the perturbative two-point functions and then fix the unknown coefficients using known values of anomalous dimensions. The same procedure can be used to determine the form of the dilation operator on the instanton moduli space. We shall denote the latter by $\tilde{D}_{(K,m)}$ to distinguish it from its counterpart acting on the space of fields. From our analysis of two-point functions it is clear that $\tilde{D}_{(K,m)}$ contains eight derivatives with respect to the variables ζ^A . In the one-instanton sector it also involves derivatives with respect to ν^A and $\bar{\nu}^A$, which can be replaced by derivatives with respect to the angular variables Ω^{AB} . We can argue that $\tilde{D}_{(1,m)}$ must be of the form

$$\tilde{D}_{(1,m)} \sim g_{\text{YM}}^8 c(N) \mathcal{A}(x_0, \rho) t_N^{A_1 B_1 \dots A_m B_m} \frac{\delta^8}{\delta(\zeta^1)^2 \delta(\zeta^2)^2 \delta(\zeta^3)^2 \delta(\zeta^4)^2} \frac{\delta^m}{\delta \Omega^{A_1 B_1} \dots \delta \Omega^{A_m B_m}}, \quad (231)$$

where the factor of g_{YM}^8 comes from the measure and the dependence on N is contained in the coefficient $c(N)$, which for large N behaves as \sqrt{N} , and in the tensor $t_N^{A_1 B_1 \dots A_m B_m}$. The latter is a projector onto SU(4) singlets. It consists of various terms corresponding to the different ways of combining $\bar{\nu}[A\nu^B]$ and $\bar{\nu}^{(A}\nu^{B)}$ bilinears to form a singlet. The N -dependence is determined by the number of SU(4) indices paired in a **6** or in a **10**. $\tilde{D}_{(1,m)}$ has also a dependence on the bosonic collective coordinates which is encoded in the function $\mathcal{A}(x_0, \rho)$. Determining the exact form of $\tilde{D}_{(1,m)}$ appears to be feasible, but more data on instanton induced anomalous dimensions are needed.

We hope to investigate further this issue in the future. In view of the previous discussion this requires the study of operators of larger dimension. The results obtained in this paper mostly show non-renormalisation properties for a large class of operators, which however are expected to be specific to small values of Δ_0 . For greater values of the bare dimension the situation is more involved because of the larger number of operators appearing. From the point of view of instanton calculations there is however a simplification since as Δ_0 grows fewer terms in the iterative solution for the elementary fields are required in semiclassical approximation. Moreover as previously remarked further simplifications arise when imposing the constraints of PSU(2,2|4), which imply that all operators in a same multiplet have equal anomalous dimension.

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A Conventions and useful relations

In this appendix we summarise the notation used in the paper and collect some useful relations.

Lower case Latin letters, i, j, k, \dots are used for the **6** of $SU(4)$ and capital letters, A, B, C, \dots for the **4**. $SO(4)$ Lorentz spinor and vector indices are indicated by Greek letters respectively from the beginning, $\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots$, and the mid, μ, ν, \dots , of the alphabet. $SU(N)$ colour indices are denoted by Latin letters from the end of the alphabet, r, s, u, v, \dots .

The $\mathcal{N}=4$ multiplet comprises six real scalars, φ^i , four Weyl fermions, λ_α^A , and a vector, A_μ with field strength $F_{\mu\nu}$, all transforming in the adjoint representation of the gauge group. It is often convenient to label the scalars by an antisymmetric pair of indices in the **4**, $\varphi^{[AB]}$, subject to the reality condition

$$\bar{\varphi}_{AB} \equiv (\varphi^{AB})^* = \frac{1}{2} \varepsilon_{ABCD} \varphi^{CD}. \quad (232)$$

In some situations the $\mathcal{N}=1$ formulation proves very useful. The $\mathcal{N}=1$ decomposition of the $\mathcal{N}=4$ supermultiplet consists of three chiral multiplets and one vector multiplet and under this decomposition only a $SU(3) \times U(1)$ subgroup of the $SU(4)$ R-symmetry group is manifest. The six scalars are combined into three complex fields, ϕ^I , $I = 1, 2, 3$, according to

$$\begin{aligned} \phi^I &= \frac{1}{\sqrt{2}} (\varphi^I + i\varphi^{I+3}) \\ \phi_I^\dagger &= \frac{1}{\sqrt{2}} (\varphi^I - i\varphi^{I+3}). \end{aligned} \quad (233)$$

The complex scalars ϕ^I and ϕ_I^\dagger transform respectively in the **3**₁ and **3**₋₁ of $SU(3) \times U(1)$. The fermions in the chiral multiplets are

$$\psi_\alpha^I = \lambda_\alpha^I, \quad \bar{\psi}_I^{\dot{\alpha}} = \bar{\lambda}_I^{\dot{\alpha}}, \quad I = 1, 2, 3, \quad (234)$$

transforming in the **3**_{3/2} and **3**_{-3/2}. The fourth fermion and the vector form the $\mathcal{N}=1$ vector multiplet, $\{\lambda_\alpha^4 = \lambda_\alpha^4, A_\mu\}$, and are $SU(3) \times U(1)$ singlets.

The two parametrisations of the $\mathcal{N}=4$ scalars, φ^i and φ^{AB} , are related by

$$\varphi^i = \frac{1}{\sqrt{2}} \Sigma_{AB}^i \varphi^{AB}, \quad \varphi^{AB} = \frac{1}{\sqrt{8}} \bar{\Sigma}_i^{AB} \varphi^i, \quad (235)$$

where Σ_{AB}^i ($\bar{\Sigma}_i^{AB}$) are Clebsch–Gordan coefficients projecting the product of two **4**'s (**4**'s) onto the **6**. These are in other words six-dimensional euclidean sigma matrices. They are defined as

$$\begin{aligned} \Sigma_{AB}^i &= (\Sigma_{AB}^a, \Sigma_{AB}^{a+3}) = (\eta_{AB}^a, i\bar{\eta}_{AB}^a) \\ \bar{\Sigma}_i^{AB} &= (\bar{\Sigma}_{AB}^a, \bar{\Sigma}_{AB}^{a+3}) = (-\eta_a^{AB}, i\bar{\eta}_a^{AB}), \end{aligned} \quad (236)$$

where $a = 1, 2, 3$ and the 't Hooft symbols η_{AB}^a and $\bar{\eta}_{AB}^a$ are

$$\begin{aligned}\eta_{AB}^a &= \bar{\eta}_{AB}^a = \varepsilon_{aAB}, & A, B &= 1, 2, 3, \\ \eta_{A4}^a &= \bar{\eta}_{4A}^a = \delta_A^a, \\ \eta_{AB}^a &= -\eta_{BA}^a, & \bar{\eta}_{AB}^a &= -\bar{\eta}_{BA}^a.\end{aligned}\tag{237}$$

Using these definitions we find the following relations among the scalars in the different formulations

$$\begin{aligned}\varphi^1 &= \frac{\sqrt{2}}{2}(\varphi^{14} + \varphi^{23}), & \varphi^2 &= \frac{\sqrt{2}}{2}(-\varphi^{13} + \varphi^{24}), & \varphi^3 &= \frac{\sqrt{2}}{2}(\varphi^{12} + \varphi^{34}), \\ \varphi^4 &= \frac{\sqrt{2}}{2}(-\varphi^{14} + \varphi^{23}), & \varphi^5 &= \frac{\sqrt{2}}{2}(-\varphi^{13} - \varphi^{24}), & \varphi^6 &= \frac{\sqrt{2}}{2}(\varphi^{12} - \varphi^{34})\end{aligned}\tag{238}$$

and

$$\begin{aligned}\phi^1 &= 2\varphi^{14}, & \phi^2 &= 2\varphi^{24}, & \phi^3 &= 2\varphi^{34}, \\ \phi_1^\dagger &= 2\varphi^{23}, & \phi_2^\dagger &= -2\varphi^{13}, & \phi_3^\dagger &= 2\varphi^{12}.\end{aligned}\tag{239}$$

The following properties of the six dimensional sigma matrices are of use

$$\varepsilon^{ABCD}\Sigma_{CD}^i = -2\bar{\Sigma}^{iAB}, \quad \varepsilon_{ABCD}\bar{\Sigma}^{iCD} = -2\Sigma_{AB}^i,\tag{240}$$

$$\Sigma_{AC}^i \bar{\Sigma}^{jCB} = \delta^{ij}\delta_A^B + \Sigma_A^{ijB},\tag{241}$$

where $\Sigma_A^{ijB} = \frac{1}{2}(\Sigma^{iAC}\bar{\Sigma}_{CB}^j - \Sigma^{jAC}\bar{\Sigma}_{CB}^i)$,

$$\bar{\Sigma}^{iAB}\Sigma_{CD}^i = 2(\delta_D^A\delta_C^B - \delta_C^A\delta_D^B), \quad \Sigma_{AB}^i\Sigma_{CD}^i = 2\varepsilon_{ABCD},\tag{242}$$

$$\Sigma_A^{ijB}\Sigma_B^{klA} = -4(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}),\tag{243}$$

$$\text{tr}(\bar{\Sigma}^i\Sigma^j\bar{\Sigma}^k\Sigma^l) = 4\delta^{ij}\delta^{kl} - 4\delta^{ik}\delta^{jl} + 4\delta^{il}\delta^{jk}.\tag{244}$$

B One-instanton in $\mathcal{N}=4$ SYM

In this appendix we briefly review the ADHM description of the one-instanton sector for the $\mathcal{N}=4$ SYM theory. A comprehensive review, including the treatment of multi-instanton configurations, can be found in [44]. Here we only recall a few elements useful for the calculations presented in this paper.

The classical instanton solution is defined in terms of a $[N+2] \times [2]$ dimensional matrix $\Delta_{a\alpha;\dot{\alpha}}$ which is a linear function of the space-time coordinate $x_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu x_\mu$,

$$\Delta_{u\alpha;\dot{\alpha}} = a_{u\alpha;\dot{\alpha}} + b_{u\alpha;\dot{\alpha}}^\beta x_{\beta\dot{\alpha}},\tag{245}$$

and its conjugate,

$$\bar{\Delta}^{\dot{\alpha};u\alpha} = \bar{a}^{\dot{\alpha};u\alpha} + x^{\dot{\alpha}\beta}\bar{b}_{\beta;\dot{\alpha}}^{u\alpha}.\tag{246}$$

The ADHM gauge field is written in the form

$$(A_\mu)_{u;v} = \bar{U}_{u;}{}^{r\alpha}\partial_\mu U_{r\alpha;v},\tag{247}$$

where the complex $[N] \times [N + 2]$ matrix $U(x)$ and its hermitian conjugate $\bar{U}(x)$ satisfy

$$\begin{aligned}\bar{U}_{u;{}^{r\alpha}} U_{r\alpha;{}^v} &= \delta_u^v, \\ \bar{\Delta}^{\dot{\alpha};r\beta} U_{r\beta;{}^v} &= 0, \quad \bar{U}_{u;{}^{r\alpha}} \Delta_{r\alpha;\dot{\alpha}} = 0.\end{aligned}\tag{248}$$

Equation (247) gives a gauge configuration with self-dual field strength provided the matrices Δ and $\bar{\Delta}$ satisfy

$$\bar{\Delta}^{\dot{\alpha};r\beta} \Delta_{r\beta;\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} f^{-1}(x),\tag{249}$$

where $f(x)$ is an arbitrary function. From this relation and the definitions (245) and (246) it follows that the coefficients a and b satisfy the bosonic ADHM constraints

$$\bar{a}^{\dot{\alpha};u\alpha} a_{u\alpha;\dot{\beta}} = \frac{1}{2} \text{Tr}(\bar{a}a) \delta^{\dot{\alpha}}_{\dot{\beta}},\tag{250}$$

$$\bar{a}^{\dot{\alpha};u\alpha} b_{u\alpha;\dot{\beta}} = \epsilon^{\beta\gamma} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{b}_{\gamma;{}^{u\alpha}} a_{u\alpha;\dot{\beta}},\tag{251}$$

$$\bar{b}_{\beta;{}^{u\alpha}} b_{u\alpha;\gamma} = \frac{1}{2} \text{Tr}(\bar{b}b) \delta^{\gamma}_{\beta}.\tag{252}$$

A choice of special frame allows to put the bosonic parameters in the form

$$b_{u\alpha;\dot{\beta}} = \begin{pmatrix} 0_{u;\dot{\beta}} \\ \delta_{\alpha}^{\dot{\beta}} \end{pmatrix}, \quad \bar{b}_{\alpha;{}^{u\beta}} = \begin{pmatrix} 0_{\alpha;{}^u} \\ \delta_{\alpha}^{\beta} \end{pmatrix},\tag{253}$$

$$a_{u\alpha;\dot{\alpha}} = \begin{pmatrix} w_{u;\dot{\alpha}} \\ a'_{\alpha\dot{\alpha}} \end{pmatrix}, \quad \bar{a}^{\dot{\alpha};u\alpha} = (\bar{w}^{\dot{\alpha};u}, \bar{a}'^{\dot{\alpha}\alpha}),\tag{254}$$

where $a'_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} a'_{\mu}$ and the components satisfy the matrix constraints

$$\text{Tr}_2(\tau^c \bar{a}a) = 0, \quad (a'_{\mu})^* = a'_{\mu}.\tag{255}$$

These ADHM collective coordinates can be easily related to the usual variables describing the position, size and gauge orientation of the instanton. The second equation in (255) expressing the reality of a' allows to identify it with the instanton position,

$$a'_{\mu} = -(x_0)_{\mu}.\tag{256}$$

Using the first equation in (255) the scale size of the instanton ρ is related to the ADHM variables by

$$\bar{w}^{\dot{\alpha}u} w_{u\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \rho^2.\tag{257}$$

The bosonic coordinates $w_{u;\dot{\alpha}}$ and $\bar{w}^{\dot{\alpha};u}$ parametrise the embedding of an $\text{SU}(2)$ instanton into $\text{SU}(N)$. We start with the special embedding

$$A_{\mu} = \begin{pmatrix} 0 & 0 \\ 0 & (A_{\mu})^{\text{SU}(2)} \end{pmatrix},\tag{258}$$

where $(A_{\mu})^{\text{SU}(2)}$ is the standard $\text{SU}(2)$ instanton solution

$$(A_{\mu})^{\text{SU}(2)} = \frac{\rho^2 \eta_{\mu\nu}^a (x - x_0)^{\nu} (\tau^a)_{\alpha}^{\beta}}{(x - x_0)^2 [(x - x_0)^2 + \rho^2]}.\tag{259}$$

The general $SU(N)$ configuration is then given by

$$A_\mu = \mathcal{H} \begin{pmatrix} 0 & 0 \\ 0 & (A_\mu)^{SU(2)} \end{pmatrix} \mathcal{H}^\dagger, \quad (260)$$

where

$$w_{u\dot{\alpha}} = \rho \mathcal{H} \begin{pmatrix} 0_{[N-2] \times [2]} \\ \mathbb{1}_{[2] \times [2]} \end{pmatrix}, \quad \mathcal{H} \in \frac{SU(N)}{SU(N-2) \times U(1)}. \quad (261)$$

The fermionic collective coordinates enter as $[N+2] \times [1]$ grassmann valued matrices, \mathcal{M} and $\bar{\mathcal{M}}$, that satisfy the ADHM constraints

$$\bar{\mathcal{M}}^{A u \alpha} a_{u\alpha; \dot{\alpha}} = -\epsilon_{\dot{\alpha}\dot{\beta}} \bar{a}^{\dot{\beta}; u \alpha} \mathcal{M}_{u\alpha}^A, \quad \bar{\mathcal{M}}^{A u \alpha} b_{u\alpha; \beta} = \epsilon^{\beta\gamma} \bar{b}_{\gamma; u \alpha} \mathcal{M}_{u\alpha}^A. \quad (262)$$

The fermionic matrices can be parametrised by

$$\mathcal{M}_{u\alpha}^A = \begin{pmatrix} \nu_u^A + w_{u; \dot{\alpha}} \mu^{\dot{\alpha} A} \\ \mathcal{M}'_{\alpha}{}^A \end{pmatrix} \equiv \begin{pmatrix} \nu_u^A + 4w_{u; \dot{\alpha}} \bar{\xi}^{\dot{\alpha} A} \\ 4\eta_{\alpha}^A \end{pmatrix}, \quad (263)$$

$$\bar{\mathcal{M}}^{A u \alpha} = (\bar{\nu}^{A u} + \bar{\mu}_{\dot{\alpha}}^A \bar{w}^{\dot{\alpha} u}, \bar{\mathcal{M}}'^{A \alpha}) \equiv (-4\bar{\xi}_{\dot{\alpha}}^A \bar{w}^{\dot{\alpha} u} + \bar{\nu}^{A u}, 4\eta^{\alpha A}), \quad (264)$$

where the ADHM conditions (262) have been used to eliminate $\bar{\mu}_{\dot{\alpha}}^A$ and $\bar{\mathcal{M}}'^{A \alpha}$ in flavour of the others and the variables ν_u^A and $\bar{\nu}^{A u}$ satisfy

$$\bar{w}^{\dot{\alpha}; u} \nu_u^A = \bar{\nu}^{A u} w_{u; \dot{\alpha}} = 0. \quad (265)$$

The sixteen fermionic collective coordinates η_{α}^A and $\bar{\xi}_{\dot{\alpha}}^A$ are identified with the zero modes associated respectively with the Poincaré and special supersymmetries broken by the bosonic instanton solution. The coordinates ν_u^A and $\bar{\nu}^{A u}$, whose total number is $8(N-2)$ because of the constraints (265), are the fermionic partners of the coset variables $w_{u; \dot{\alpha}}$ and $\bar{w}^{\dot{\alpha}; u}$ parametrising the gauge orientations.

We now summarise the expressions for the leading order terms in the iterative solution of the field equations for the elementary fields in the $\mathcal{N}=4$ SYM multiplet as given in terms of the ADHM variables. All the fields in the multiplet are in the adjoint representation of the gauge group $SU(N)$ and can be represented as $[N] \times [N]$ matrices.

The (self-dual part of the) gauge field strength $F_{\mu\nu}^{(0)-}$ in the instanton background follows from the construction of the gauge field A_μ in the previous section. The result is

$$\left(F^{(0)\mu\nu} \right)_{u; v} = \bar{U}_{u; r\alpha} b_{r\alpha; \beta} \sigma^{\mu\nu}{}_{\beta}{}^{\gamma} f \bar{b}_{\gamma; s\delta} U_{s\delta; v}. \quad (266)$$

The Weyl fermions $\lambda_{\alpha}^{(1)A}$ solve the equation

$$\mathcal{D}^{(0)} \lambda_{\alpha}^{(1)A} = 0 \quad (267)$$

and in terms of the ADHM variables can be written as

$$\left(\lambda_{\alpha}^{(1)A} \right)_{u; v} = \bar{U}_{u; r\beta} \left(\mathcal{M}_{r\beta}^A f \bar{b}_{\alpha; s\gamma} - \varepsilon_{\alpha\delta} b_{r\beta; \delta} f \bar{\mathcal{M}}^{As\gamma} \right) U_{s\gamma; v}. \quad (268)$$

Analogously the term $\varphi^{(2)AB}$ in the the solution for the scalar field is determined by

$$\mathcal{D}^2 \varphi^{(2)AB} = \frac{i}{\sqrt{2}} \{ \lambda^{(1)A}, \lambda^{(1)B} \}. \quad (269)$$

The solution was constructed in [45]. In the $\mathcal{N}=4$ SYM theory in the superconformal phase of interest here it can be written in the form

$$\begin{aligned} \varphi^{(2)AB} &= \frac{1}{2} \bar{U}_{u; \ r\alpha} \left(\mathcal{M}_{r\alpha}^B f \bar{\mathcal{M}}^{As\beta} - \mathcal{M}_{r\alpha}^A f \bar{\mathcal{M}}^{Bs\beta} \right) U_{s\gamma; \ v} \\ &+ \frac{1}{2} \bar{U}_{u; \ r\alpha} \begin{pmatrix} 0_{r; \ s} & 0_{r; \ \beta} \\ 0_{\alpha; \ s} & \mathcal{A}^{AB} \delta_{\alpha}^{\beta} \end{pmatrix} U_{s\beta; \ v}, \end{aligned} \quad (270)$$

where \mathcal{A}^{AB} is defined by

$$\mathcal{A}^{AB} = \frac{1}{2\sqrt{2}\rho^2} \left(\bar{\mathcal{M}}^{Ar\alpha} \mathcal{M}_{r\alpha}^B - \bar{\mathcal{M}}^{Br\alpha} \mathcal{M}_{r\alpha}^A \right). \quad (271)$$

The gauge invariant composite operators we are interested in are traces over colour indices of products of elementary fields. In evaluating correlation functions of such operators in the semiclassical approximation in the instanton background one must then compute expressions of the form

$$\mathcal{O} = \text{Tr}_N \left(\bar{U} \tilde{F} U \dots \bar{U} \tilde{\lambda} U \dots \bar{U} \tilde{\varphi} U \dots \right). \quad (272)$$

It is convenient to rewrite such expressions as traces over $[N+2] \times [N+2]$ matrices in the following way

$$\begin{aligned} \mathcal{O} &= \text{Tr}_N \left(\bar{U} \tilde{F} U \dots \bar{U} \tilde{\lambda} U \dots \bar{U} \tilde{\varphi} U \dots \right) \\ &= \text{Tr}_{N+2} \left[(\mathcal{P} \tilde{F}) \dots (\mathcal{P} \tilde{\lambda}) \dots (\mathcal{P} \tilde{\varphi}) \dots \right], \end{aligned} \quad (273)$$

where we have defined the projection operator

$$\mathcal{P}_{u\alpha; \ v\beta} = U_{u\alpha; \ r} \bar{U}_{r; \ v\beta} = \delta_{u; \ v} \delta_{\alpha}^{\beta} - \Delta_{u\alpha; \ \gamma} f \bar{\Delta}_{\gamma; \ v\beta}, \quad (274)$$

where Δ and $\bar{\Delta}$ are the matrices of bosonic ADHM variables defined in (245) and (246).

In the one-instanton sector all the previous formulae can be made very explicit. The function $f(x)$ entering the ADHM construction is

$$f = f(x; x_0, \rho) = \frac{1}{(x - x_0)^2 + \rho^2} = \frac{1}{y^2 + \rho^2}. \quad (275)$$

and the projector \mathcal{P} becomes

$$\mathcal{P}_{u\alpha; \ v\beta} = \delta_{u; \ v} \delta_{\alpha}^{\beta} - \frac{1}{y^2 + \rho^2} \begin{pmatrix} w_{u; \dot{\alpha}} \bar{w}^{\dot{\alpha}; v} & w_{u; \dot{\alpha}} y^{\dot{\alpha}\beta} \\ y_{\alpha\dot{\alpha}} \bar{w}^{\dot{\alpha}; v} & y^2 \delta_{\alpha}^{\beta} \end{pmatrix}. \quad (276)$$

Notice in particular that it satisfies

$$\text{Tr}_{N+2} [\mathcal{P}(x)] = N. \quad (277)$$

As observed above the gauge invariant composite operators can be expressed as traces over $[N+2] \times [N+2]$ matrices. In the following we give the formulae for the ‘projected’ $[N+2]$ -dimensional matrices corresponding to the solutions (266), (268) and (270) for the elementary fields. For this purpose we introduce the notation for a generic Yang–Mills field Φ

$$\begin{aligned}\Phi_{u;v} &= \bar{U}_{u;}{}^{r\beta} \tilde{\Phi}_{r\beta; s\gamma} U_{s\gamma;v} \\ \hat{\Phi}_{u\alpha;v\beta} &= \mathcal{P}_{u\alpha;}{}^{r\gamma} \tilde{\Phi}_{r\gamma;v\beta}.\end{aligned}\quad (278)$$

For the field strength $F_{\mu\nu}$ we get ⁸

$$(\hat{F}_{\mu\nu})_{u\alpha;v\beta} = \begin{pmatrix} 0 & \left(\hat{F}_{\mu\nu}^{(1)}\right)_{u;v}{}^{\beta} \\ 0 & \left(\hat{F}_{\mu\nu}^{(2)}\right)_{\alpha;v}{}^{\beta} \end{pmatrix}, \quad (279)$$

where

$$\left(\hat{F}_{mn}^{(1)}\right)_{u;v}{}^{\beta} = -\frac{4}{(y^2 + \rho^2)^2} w_{u;\dot{\alpha}} y^{\dot{\alpha}\gamma} \sigma_{mn\gamma}{}^{\beta}, \quad \left(\hat{F}_{mn}^{(2)}\right)_{\alpha;v}{}^{\beta} = \frac{4}{(y^2 + \rho^2)^2} \rho^2 \sigma_{mn\alpha}{}^{\beta}. \quad (280)$$

The solution for the fermions λ_{α}^A is

$$(\hat{\lambda}_{\alpha}^A)_{u\beta;v\gamma} = \begin{pmatrix} \left(\hat{\lambda}_{\alpha}^{(1)A}\right)_{u;v}{}^{\gamma} & \left(\hat{\lambda}_{\alpha}^{(2)A}\right)_{u;v}{}^{\gamma} \\ \left(\hat{\lambda}_{\alpha}^{(3)A}\right)_{\beta;v}{}^{\gamma} & \left(\hat{\lambda}_{\alpha}^{(4)A}\right)_{\beta;v}{}^{\gamma} \end{pmatrix}, \quad (281)$$

where

$$\begin{aligned}\left(\hat{\lambda}_{\alpha}^{(1)A}\right)_{u;v}{}^{\gamma} &= \frac{1}{(y^2 + \rho^2)^2} \varepsilon_{\alpha\delta} w_{u;\dot{\alpha}} y^{\dot{\alpha}\delta} (-4\bar{\xi}_{\beta}^A \bar{w}^{\dot{\beta}v} + \bar{\nu}^A v) \\ \left(\hat{\lambda}_{\alpha}^{(2)A}\right)_{u;v}{}^{\gamma} &= \frac{1}{(y^2 + \rho^2)^2} \left[4 \left(y^2 w_{u;\dot{\alpha}} \bar{\xi}^{\dot{\alpha}A} \delta_{\alpha}^{\gamma} - w_{u;\dot{\alpha}} y^{\dot{\alpha}\delta} \eta_{\delta}^A \delta_{\alpha}^{\gamma} + \varepsilon_{\alpha\delta} w_{u;\dot{\alpha}} y^{\dot{\alpha}\delta} \eta^{\gamma A} \right) \right. \\ &\quad \left. + (y^2 + \rho^2) \nu_u^A \delta_{\alpha}^{\gamma} \right] \\ \left(\hat{\lambda}_{\alpha}^{(3)A}\right)_{\beta;v}{}^{\gamma} &= \frac{\rho^2}{(y^2 + \rho^2)^2} \varepsilon_{\alpha\beta} (4\bar{\xi}_{\dot{\alpha}}^A \bar{w}^{\dot{\alpha}v} - \bar{\nu}^A v) \\ \left(\hat{\lambda}_{\alpha}^{(4)A}\right)_{\beta;v}{}^{\gamma} &= \frac{4\rho^2}{(y^2 + \rho^2)^2} \left(-y_{\beta\dot{\alpha}} \bar{\xi}^{\dot{\alpha}A} \delta_{\alpha}^{\gamma} + \eta_{\beta}^A \delta_{\alpha}^{\gamma} - \varepsilon_{\alpha\beta} \eta^{\gamma A} \right).\end{aligned}\quad (282)$$

Finally the scalar field φ^{AB} reads

$$(\hat{\varphi}^{AB})_{u\beta;v\gamma} = \begin{pmatrix} \left(\hat{\varphi}^{(1)AB}\right)_{u;v}{}^{\gamma} & \left(\hat{\varphi}^{(2)AB}\right)_{u;v}{}^{\gamma} \\ \left(\hat{\varphi}^{(3)AB}\right)_{\beta;v}{}^{\gamma} & \left(\hat{\varphi}^{(4)AB}\right)_{\beta;v}{}^{\gamma} \end{pmatrix}, \quad (283)$$

⁸In the remainder of this appendix we omit the superscript denoting the number of fermion modes on the fields. Superscripts in parenthesis refer here to matrix elements.

where

$$\begin{aligned}
(\hat{\varphi}^{(1)AB})_{u; \quad}^v &= \frac{1}{4(y^2 + \rho^2)^2} \left\{ y^2 \left[-16(\bar{\xi}^{\dot{\alpha}B} \bar{\xi}_{\dot{\beta}}^A - \bar{\xi}^{\dot{\alpha}A} \bar{\xi}_{\dot{\beta}}^B) w_{u;\dot{\alpha}} \bar{w}^{\dot{\beta};v} \right. \right. \\
&\quad + 4w_{u;\dot{\alpha}} (\bar{\xi}^{\dot{\alpha}B} \bar{\nu}^{Av} - \bar{\xi}^{\dot{\alpha}A} \bar{\nu}^{Bv}) \left. \right] \\
&\quad + (y^2 + \rho^2) \left[-4(\bar{\xi}_{\dot{\alpha}}^B \nu_u^A - \bar{\xi}_{\dot{\alpha}}^A \nu_u^B) \bar{w}^{\dot{\alpha};v} + (\nu_u^B \bar{\nu}^{Av} - \nu_u^A \bar{\nu}^{Bv}) \right] \\
&\quad + y^{\dot{\alpha}\delta} \left[16(\eta_{\delta}^B \bar{\xi}_{\dot{\beta}}^A - \eta_{\delta}^A \bar{\xi}_{\dot{\beta}}^B) w_{u;\dot{\alpha}} \bar{w}^{\dot{\beta};v} - 4w_{u;\dot{\alpha}} (\eta_{\delta}^B \bar{\nu}^{Av} - \eta_{\delta}^A \bar{\nu}^{Bv}) \right] \left. \right\} \\
(\hat{\varphi}^{(2)AB})_{u; \quad}^{\gamma} &= \frac{1}{4(y^2 + \rho^2)^2} \left\{ 16y^2 w_{u;\dot{\alpha}} (\bar{\xi}^{\dot{\alpha}B} \eta^{\gamma A} - \bar{\xi}^{\dot{\alpha}A} \eta^{\gamma B}) + 4(y^2 + \rho^2) (\nu_u^B \eta^{\gamma A} - \nu_u^A \eta^{\gamma B}) \right. \\
&\quad - w_{u;\dot{\alpha}} \left[16y^{\dot{\alpha}\delta} (\eta_{\delta}^B \eta^{\gamma A} - \eta_{\delta}^A \eta^{\gamma B}) + \frac{1}{2} \frac{y^2 + \rho^2}{\rho^2} y^{\dot{\alpha}\gamma} (\bar{\nu}^{Au} \nu_u^B - \bar{\nu}^{Br} \nu_r^A) \right] \left. \right\} \\
(\hat{\varphi}^{(3)AB})_{\beta; \quad}^v &= \frac{1}{4(y^2 + \rho^2)^2} \left\{ \rho^2 \left[16y_{\beta\dot{\alpha}} (\bar{\xi}^{\dot{\alpha}B} \bar{\xi}_{\dot{\beta}}^A - \bar{\xi}^{\dot{\alpha}A} \bar{\xi}_{\dot{\beta}}^B) \bar{w}^{\dot{\beta};v} - 4y_{\beta\dot{\alpha}} (\bar{\xi}^{\dot{\alpha}B} \bar{\nu}^{Av} - \bar{\xi}^{\dot{\alpha}A} \bar{\nu}^{Bv}) \right. \right. \\
&\quad - 16(\eta_{\beta}^B \bar{\xi}_{\dot{\alpha}}^A - \eta_{\beta}^A \bar{\xi}_{\dot{\alpha}}^B) \bar{w}^{\dot{\alpha};v} + 4(\eta_{\beta}^B \bar{\nu}^{Av} - \eta_{\beta}^A \bar{\nu}^{Bv}) \left. \right] \left. \right\} \\
(\hat{\varphi}^{(4)AB})_{\beta; \quad}^{\gamma} &= \frac{\rho^2}{4(y^2 + \rho^2)^2} \left[-16y_{\beta\dot{\alpha}} (\bar{\xi}^{\dot{\alpha}B} \eta^{\gamma A} - \bar{\xi}^{\dot{\alpha}A} \eta^{\gamma B}) + 16(\eta_{\beta}^B \eta^{\gamma A} - \eta_{\beta}^A \eta^{\gamma B}) \right. \\
&\quad + \left. \frac{1}{2} \frac{y^2 + \rho^2}{\rho^2} \delta_{\beta}^{\gamma} (\bar{\nu}^{Ar} \nu_r^B - \bar{\nu}^{Br} \nu_r^A) \right]. \tag{284}
\end{aligned}$$

In the calculation of correlation functions at leading order in g_{YM} in non-trivial topological sectors we need the expression for the propagators in the instanton background. The propagator for the adjoint scalars in the one-instanton background takes the form

$$\begin{aligned}
\langle \tilde{\varphi}_{r\alpha}^{AB u\gamma}(x) \tilde{\varphi}_{s\beta}^{CD v\delta}(y) \rangle &\equiv \tilde{G}^{ABCD} [{}^{u\gamma, v\delta}_{r\alpha, s\beta}](x, y) \\
&= \frac{g_{\text{YM}}^2 \varepsilon^{ABCD}}{2\pi^2 (x-y)^2} \left\{ [\mathcal{P}(x) \mathcal{P}(y)]_{r\alpha}{}^{v\delta} [\mathcal{P}(y) \mathcal{P}(x)]_{s\beta}{}^{u\gamma} \right. \\
&\quad - \frac{1}{N} \left([\mathcal{P}(x)]_{r\alpha}{}^{u\gamma} [\mathcal{P}(y) \mathcal{P}(x) \mathcal{P}(y)]_{s\beta}{}^{v\delta} + [\mathcal{P}(y)]_{s\beta}{}^{v\delta} [\mathcal{P}(x) \mathcal{P}(y) \mathcal{P}(x)]_{r\alpha}{}^{u\gamma} \right) \\
&\quad + \frac{1}{N^2} [\mathcal{P}(x)]_{r\alpha}{}^{u\gamma} [\mathcal{P}(y)]_{s\beta}{}^{v\delta} \text{Tr}_{N+2} [\mathcal{P}(x) \mathcal{P}(y)] \left. \right\} \\
&\quad + \frac{g_{\text{YM}}^2 \varepsilon^{ABCD}}{4\pi^2 \rho^2} \left\{ [\mathcal{P}(x) b \bar{b} \mathcal{P}(x)]_{r\alpha}{}^{u\gamma} [\mathcal{P}(y) b \bar{b} \mathcal{P}(y)]_{s\beta}{}^{v\delta} \right. \\
&\quad - \frac{1}{N} \left([\mathcal{P}(x)]_{r\alpha}{}^{u\gamma} [\mathcal{P}(y) b \bar{b} \mathcal{P}(y)]_{s\beta}{}^{v\delta} \text{Tr}_2 [\bar{b} \mathcal{P}(x) b] \right. \\
&\quad + [\mathcal{P}(y)]_{s\beta}{}^{v\delta} [\mathcal{P}(x) b \bar{b} \mathcal{P}(x)]_{r\alpha}{}^{u\gamma} \text{Tr}_2 [\bar{b} \mathcal{P}(y) b] \left. \right) \\
&\quad + \frac{1}{N^2} [\mathcal{P}(x)]_{r\alpha}{}^{u\gamma} [\mathcal{P}(y)]_{s\beta}{}^{v\delta} \text{Tr}_2 [\bar{b} \mathcal{P}(x) b] \text{Tr}_2 [\bar{b} \mathcal{P}(y) b] \left. \right\}. \tag{285}
\end{aligned}$$

The propagators for the fermions, $\langle \bar{\lambda}_{A\dot{\alpha}} \lambda_{\alpha}^B \rangle$, and for the vector, $\langle A_{\mu} A_{\nu} \rangle$, can be deduced from the scalar Green function [47].

C OPE analysis of a four-point function

In this appendix we present the calculation of the four-point function (93),

$$G(x_1, x_2, x_3, x_4) = \langle \mathcal{Q}^{i_1 j_1 k_1}(x_1) \mathcal{Q}^{i_2 j_2}(x_2) \mathcal{Q}^{i_3 j_3 k_3}(x_3) \mathcal{Q}^{i_4 j_4}(x_4) \rangle. \quad (286)$$

As discussed in section 5.2.5 the double pinching limit, $x_{12} \rightarrow 0$, $x_{34} \rightarrow 0$, allows to extract the anomalous dimension of the $\Delta_0 = 3$ operator in the **6** of SU(4) from its contribution to the OPE. The computation of the four-point function can be drastically simplified by a suitable choice of components in (286). It is convenient to work with complex fields in the $\mathcal{N}=1$ formulation. We use the SU(4) \rightarrow SU(3) \times U(1) branching rules

$$\mathbf{20}' \rightarrow \mathbf{6}_2 \oplus \bar{\mathbf{6}}_{-2} \oplus \mathbf{8}_0 \quad (287)$$

$$\mathbf{50} \rightarrow \mathbf{10}_3 \oplus \bar{\mathbf{10}}_{-3} \oplus \mathbf{15}_1 \oplus \bar{\mathbf{15}}_{-1}, \quad (288)$$

where the subscript denotes the U(1) charge. Under this decomposition the operators in the $\mathbf{20}'$ and $\mathbf{50}$, \mathcal{Q}^{ij} and \mathcal{Q}^{ijk} , decompose respectively into

$$\mathcal{C}^{IJ} = \frac{1}{g_{\text{YM}}^2} \text{Tr}(\phi^I \phi^J) \in \mathbf{6}_2 \quad (289)$$

$$\bar{\mathcal{C}}_{IJ} = \frac{1}{g_{\text{YM}}^2} \text{Tr}(\phi_I^\dagger \phi_J^\dagger) \in \bar{\mathbf{6}}_{-2} \quad (290)$$

$$\mathcal{V}_I^J = \frac{1}{g_{\text{YM}}^2} \text{Tr}(\phi_I^\dagger \phi^J) - \frac{1}{3g_{\text{YM}}^2} \delta_I^J \text{Tr}(\phi_K^\dagger \phi^K) \in \mathbf{8}_0 \quad (291)$$

and

$$\mathcal{C}^{IJK} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr}(\phi^I \phi^J \phi^K + \phi^I \phi^K \phi^J) \in \mathbf{10}_3 \quad (292)$$

$$\bar{\mathcal{C}}_{IJK} = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr}(\phi_I^\dagger \phi_J^\dagger \phi_K^\dagger + \phi_I^\dagger \phi_K^\dagger \phi_J^\dagger) \in \bar{\mathbf{10}}_{-3} \quad (293)$$

$$\begin{aligned} \mathcal{V}_I^{JK} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr}(\phi_I^\dagger \phi^J \phi^K + \phi_I^\dagger \phi^K \phi^J) - \frac{1}{4g_{\text{YM}}^3 N^{1/2}} [\delta_I^J \text{Tr}(\phi_L^\dagger \phi^L \phi^K + \phi_L^\dagger \phi^K \phi^L) \\ &\quad + \delta_I^K \text{Tr}(\phi_L^\dagger \phi^L \phi^J + \phi_L^\dagger \phi^J \phi^L)] \in \mathbf{15}_1 \end{aligned} \quad (294)$$

$$\begin{aligned} \bar{\mathcal{V}}^I_{JK} &= \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr}(\phi^I \phi_J^\dagger \phi_K^\dagger + \phi^I \phi_K^\dagger \phi_J^\dagger) - \frac{1}{4g_{\text{YM}}^3 N^{1/2}} [\delta^I_J \text{Tr}(\phi^L \phi_L^\dagger \phi_K^\dagger + \phi^L \phi_K^\dagger \phi_L^\dagger) \\ &\quad + \delta^I_K \text{Tr}(\phi^L \phi_L^\dagger \phi_J^\dagger + \phi^L \phi_J^\dagger \phi_L^\dagger)] \in \bar{\mathbf{15}}_{-1}. \end{aligned} \quad (295)$$

A simple choice of components, which leads to a contribution of the **6** in the $x_{12} \rightarrow 0$ $x_{34} \rightarrow 0$ channel, is then

$$G(x_1, x_2, x_3, x_4) = \langle \mathcal{C}^{113}(x_1) \bar{\mathcal{C}}_{11}(x_2) \mathcal{V}_3^{11}(x_3) \bar{\mathcal{C}}_{11}(x_4) \rangle. \quad (296)$$

Recalling the relation between the complex scalars ϕ^I and the φ^{AB} 's we find

$$\mathcal{C}^{113} \sim \text{Tr}(\varphi^{14} \varphi^{14} \varphi^{34}), \quad \bar{\mathcal{C}}_{11} \sim \text{Tr}(\varphi^{23} \varphi^{23}), \quad \mathcal{V}_3^{11} \sim \text{Tr}(\varphi^{12} \varphi^{14} \varphi^{14}), \quad (297)$$

which shows that (296) can saturate the sixteen superconformal modes.

In the one-instanton sector this correlation function receives two types of contributions. The operators in (296) contain at least twenty fermionic modes. Therefore one can either replace all the fields by their classical instanton solutions, in which case four of the fermion modes have to be of the ν and $\bar{\nu}$ type, or contract one pair of scalars.

For the first type of contribution substituting the classical expressions for the composite fields and after some simple Fierz rearrangements we get

$$\begin{aligned}
G^{(1)}(x_1, x_2, x_3, x_4) &= \int d\mu_{\text{phys}} e^{-S_{\text{inst}}} \hat{\mathcal{C}}^{113}(x_1; x_0, \rho; \zeta, \nu, \bar{\nu}) \hat{\mathcal{C}}_{11}(x_2; x_0, \rho; \zeta, \nu, \bar{\nu}) \\
&\quad \hat{\mathcal{V}}_3^{11}(x_3; x_0, \rho; \zeta, \nu, \bar{\nu}) \hat{\mathcal{C}}_{11}(x_4; x_0, \rho; \zeta, \nu, \bar{\nu}) \\
&= \frac{\pi^{-4N} g_{\text{YM}}^{4N-10} e^{2\pi i \tau}}{N(N-1)!(N-2)!} \int d^4 x_0 d\rho \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A d^{N-2} \nu^A d^{N-2} \bar{\nu}^A e^{-S_{4F}} \\
&\quad \frac{\rho^4 [\zeta^1(x_1) \zeta^1(x_1)] [\zeta^4(x_1) \zeta^4(x_1)] (\bar{\nu}^{[3} \nu^{4]})}{(y_1^2 + \rho^2)^5} \frac{\rho^4 [\zeta^2(x_2) \zeta^2(x_2)] [\zeta^3(x_2) \zeta^3(x_2)]}{(y_2^2 + \rho^2)^4} \\
&\quad \frac{\rho^4 [\zeta^1(x_3) \zeta^1(x_3)] [\zeta^4(x_3) \zeta^4(x_3)] (\bar{\nu}^{[1} \nu^{2]})}{(y_3^2 + \rho^2)^5} \frac{\rho^4 [\zeta^2(x_4) \zeta^2(x_4)] [\zeta^3(x_4) \zeta^3(x_4)]}{(y_4^2 + \rho^2)^4}, \tag{298}
\end{aligned}$$

where the spinor indices on pairs of ζ 's in square brackets are understood to be contracted. In (298) as usual an overall numerical constant has been omitted and is to be restored in the final result. The integrations over the superconformal fermion modes can then be performed using (42). The integrations over the ν and $\bar{\nu}$ modes can be treated similarly to what was done in previous cases and after simple manipulations one is left with five-sphere integrals of the form (69). After computing all the fermionic integrals we obtain

$$\begin{aligned}
G^{(1)}(x_1, x_2, x_3, x_4) &= \frac{3^4 \pi^{-15} 2^{-2N-15} (N^2 - 3N + 2) \Gamma(2N - 2) e^{2\pi i \tau}}{N(N-1)!(N-2)!} x_{13}^4 x_{24}^4 \\
&\quad \int d^4 x_0 d\rho \frac{\rho^{13}}{[(x_1 - x_0)^2 + \rho^2]^5 [(x_2 - x_0)^2 + \rho^2]^4 [(x_3 - x_0)^2 + \rho^2]^5 [(x_4 - x_0)^2 + \rho^2]^4}, \tag{299}
\end{aligned}$$

where $x_{pq} = (x_p - x_q)$. The final integrations over ρ and x_0 can be rewritten as

$$\begin{aligned}
&\int d^4 x_0 d\rho \frac{\rho^{13}}{[(x_1 - x_0)^2 + \rho^2]^5 [(x_2 - x_0)^2 + \rho^2]^4 [(x_3 - x_0)^2 + \rho^2]^5 [(x_4 - x_0)^2 + \rho^2]^4} \\
&= c \frac{\partial}{\partial x_{13}^2} \prod_{i < j} \frac{\partial}{\partial x_{ij}^2} B(x_1, x_2, x_3, x_4), \tag{300}
\end{aligned}$$

where c is a numerical constant and we have introduced the box integral, $B(x_1, x_2, x_3, x_4)$, defined as

$$B(x_1, x_2, x_3, x_4) = \int d^4 x \frac{1}{(x - x_1)^2 (x - x_2)^2 (x - x_3)^2 (x - x_4)^2}. \tag{301}$$

The four-point function (299) is finite and in the double limit $x_{12} \rightarrow 0$, $x_{34} \rightarrow 0$ its leading singularity is

$$G^{(1)}(x_1, x_2, x_3, x_4) \xrightarrow[x_{34} \rightarrow 0]{x_{12} \rightarrow 0} \frac{1}{x_{13}^5 x_{24}^5} \log \left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \right). \tag{302}$$

The second type of contribution involves scalar propagators. Recalling that $\langle \varphi^{AB} \varphi^{CD} \rangle \sim \varepsilon^{ABCD}$, from (296)-(297) it follows that the allowed contractions are

$$\begin{aligned} G^{(2)}(x_1, x_2, x_3, x_4) &\sim \langle \text{Tr} \left[\left(\varphi^{14} \varphi^{14} \varphi^{34} \right) (x_1) \right] \text{Tr} \left[\left(\varphi^{23} \varphi^{23} \right) (x_2) \right] \right. \\ &\quad \text{Tr} \left[\left(\varphi^{12} \varphi^{14} \varphi^{14} \right) (x_3) \right] \text{Tr} \left[\left(\varphi^{23} \varphi^{23} \right) (x_4) \right] \rangle + \dots \\ &\quad \left. + \langle \text{Tr} \left[\left(\varphi^{14} \varphi^{14} \varphi^{34} \right) (x_1) \right] \text{Tr} \left[\left(\varphi^{23} \varphi^{23} \right) (x_2) \right] \text{Tr} \left[\left(\varphi^{12} \varphi^{14} \varphi^{14} \right) (x_3) \right] \text{Tr} \left[\left(\varphi^{23} \varphi^{23} \right) (x_4) \right] \right\rangle, \end{aligned} \quad (303)$$

where the dots in (303) stand for the other contractions of the same type as those indicated as required by Wick's theorem. The first type of contraction, between a φ^{14} and a φ^{23} , leads however to a vanishing contribution since after replacing the remaining fields with their instanton solution one is forced to put three ζ^1 modes at the same point, x_3 . We are thus left with only one possible contraction, the one on the last line of (303). To evaluate the corresponding contribution to the four-point function we use the propagator (285). The calculation is rather involved and gives

$$\begin{aligned} G^{(2)}(x_1, x_2, x_3, x_4) &= \frac{3^4 \pi^{-15} 2^{-2N-13} \Gamma(2N-1) e^{2\pi i \tau}}{N(N-1)!(N-2)!} \int d^4 x_0 d\rho d^5 \Omega \prod_{A=1}^4 d^2 \eta^A d^2 \bar{\xi}^A \\ &\quad \left[\frac{4}{(x_1 - x_3)^2} \frac{\rho^8}{(y_1^2 + \rho^2)^4 (y_3^2 + \rho^2)^4} - \frac{(3N^2 + 10N + 12)}{N^2} \frac{\rho^{10}}{(y_1^2 + \rho^2)^5 (y_3^2 + \rho^2)^5} \right] \\ &\quad \frac{\rho^8}{(y_2^2 + \rho^2)^4 (y_4^2 + \rho^2)^4} \left[(\zeta^1 \zeta^1)(\zeta^4 \zeta^4)(x_1) \right] \left[(\zeta^1 \zeta^1)(\zeta^4 \zeta^4)(x_3) \right] \\ &\quad \left[(\zeta^2 \zeta^2)(\zeta^3 \zeta^3)(x_2) \right] \left[(\zeta^2 \zeta^2)(\zeta^3 \zeta^3)(x_4) \right], \end{aligned} \quad (304)$$

where the ν and $\bar{\nu}$ integrals have been replaced by a five-sphere integral since in this contribution there is no explicit dependence on these variables in the integrand. The integrations over the superconformal modes can now be performed and we get

$$\begin{aligned} G^{(2)}(x_1, x_2, x_3, x_4) &= \frac{3^4 \pi^{-15} 2^{-2N-13} \Gamma(2N-1) e^{2\pi i \tau}}{N(N-1)!(N-2)!} (x_1 - x_3)^4 (x_2 - x_4)^4 \\ &\quad \int d^4 x_0 d\rho \frac{\rho^{11}}{[(x_1 - x_0)^2 + \rho^2]^4 [(x_2 - x_0)^2 + \rho^2]^4 [(x_3 - x_0)^2 + \rho^2]^4 [(x_4 - x_0)^2 + \rho^2]^4} \\ &\quad \left[\frac{4}{(x_1 - x_3)^2} - \frac{(3N^2 + 10N + 12)}{N^2} \frac{\rho^2}{(y_1^2 + \rho^2)(y_3^2 + \rho^2)} \right], \end{aligned} \quad (305)$$

which can be rewritten in terms of the box integral (301) similarly to the case of $G^{(1)}(x_1, x_2, x_3, x_4)$. In the limit $x_{12} \rightarrow 0$ $x_{34} \rightarrow 0$ the singularity is again of the type

$$G^{(2)}(x_1, x_2, x_3, x_4) \xrightarrow[x_{34} \rightarrow 0]{x_{12} \rightarrow 0} \frac{1}{x_{13}^5 x_{24}^5} \log \left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \right). \quad (306)$$

In conclusion the OPE of the complete four-point function (296) does not present a singularity corresponding to the contribution of a scalar operator of bare dimension $\Delta_0 = 3$ in the **6** of $\text{SU}(4)$ in the $x_{12} \rightarrow 0$, $x_{34} \rightarrow 0$ channel. Therefore the operator

$$\mathcal{O}_{3,6}^i = \frac{1}{g_{\text{YM}}^3 N^{1/2}} \text{Tr} \left(\varphi^i \varphi^j \varphi^j \right). \quad (307)$$

which according to the analysis of section 5.2.1 could have an instanton induced anomalous dimension appears instead to be protected at the instanton level.

The singularities observed in (302) and (306) on the other hand correspond to the contribution of operators of bare dimension 5. Since in section 5.4 it was argued that at the level of dimension 5 operators only the ones in the **6** can have an instanton contribution, the result of the above OPE analysis confirms that indeed at least one of the $\Delta_0 = 5$ operators in the **6** has scaling dimension corrected by instantons. The study of a single four-point function however does not allow to identify which operators in this sector receive corrections.

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